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A THEORETICAL INVESTIGATION OF A CLASS
OF LINEAR MEASUREMENT SYSTEMS SUBJECT
TO RANDOM NOISE

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Approved

D. L. Finn

J. H. MacKay

W. B. Jones

Date approved by chairman May 17, 1961

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LIST OF ILLUSTRATIONS

Figure		Page
1.	Logical Block Diagram for a Class of Measurement Systems . .	4
2.	A Measurement System	9
3.	A Plot of the Minimum Value of $P(y(t_1) - m < C\sigma)$ vs $C\sigma$. .	12
4.	Schematic Representation of the Mathematical Model	24
5.	Schematic Representation of the Linear Noisy Measurement System	31
6.	Equivalent Circuit of Hot Wire Anemometer	32
7.	Flow Graphs of Hot Wire Anemometer	34
8.	Flow Graphs of Hot Wire Anemometer	37
9.	Representative Group of Elements from Biased Estimating System	39
10.	Illustration of a Scale and Pointer	59
11.	A Representative Sequence of Meter Readings	63
12.	Extreme Probability Density Functions for $r_2(t)$ for Operation in the Linear Range	65
13.	Two Possible Modes of Measurement System Operation	87
14.	A Representative Measurement System	123
15.	Block Diagram of Representative Measurement System	124
16.	Representation of Measurement System in Standard Form . . .	127
17.	$\alpha(\omega_n, \xi; t)/\omega_n$ vs $\omega_n t$	132
18.	$t\alpha(\omega_n, \xi; t)$ vs $\omega_n t$	133

TABLE OF CONTENTS

	Page
ACKNOWLEDGEMENTS	11
LIST OF ILLUSTRATIONS	iv
SUMMARY	v
 CHAPTER	
I. INTRODUCTION	1
II. FORMULATION OF A GENERAL LOGICAL STRUCTURE FOR A CLASS OF MEASUREMENT SYSTEMS	3
III. GENERAL DISCUSSION OF THE NATURE OF NOISY MEASUREMENT SYSTEM ERRORS	8
IV. FORMULATION OF A MATHEMATICAL MODEL FOR THE TRANSDUCER, SUBTRACTING DEVICE, AMPLIFIER AND REFERENCE SOURCE	16
V. ANALYSIS OF THE BIASED ESTIMATING SYSTEM	38
VI. FORMULATION AND ANALYSIS OF A MATHEMATICAL MODEL FOR THE OUTPUT METER	55
VII. FORMULATION OF GENERAL EXPRESSIONS FOR MEASUREMENT SYSTEM ERRORS	85
VIII. BIASED ESTIMATING SYSTEM ERRORS IN CERTAIN SPECIAL CASES .	102
IX. SUMMARY AND ENGINEERING APPLICATION OF RESULTS	112
X. AN EXAMPLE ILLUSTRATING THE DESIGN OF A MEASUREMENT SYSTEM	122
BIBLIOGRAPHY	137
VITA	139

SUMMARY

The purpose of this study is to take a first step in the direction of formulating a precise mathematical theory of the principles of measurement systems. In the recent literature very little work has been found which treats noisy measurement systems with deliberate generality. Thus, since this study is in a field which is not well established, the emphasis has had to be on the general formulation of the problem. A restricted, but important, class of measurement systems has been chosen and a complete mathematical model has been developed for this class of systems.

Although the majority of this work applies to only this particular restricted class of measurement systems, a general logical structure which is applicable to a much wider class of systems is presented and the nature of the errors in such systems is discussed. This discussion brings out the fact that only the statistical parameters of the output of a measurement system can be used in any practical way. Two statistical parameters identified as bias error and mean square error are defined and suggested as a means for describing measurement system performance.

The major portion of this study is devoted to the development, analysis, and discussion of a mathematical model for what is termed a "linear noisy measurement system." Three operational components are identified as being the important elements of the linear noisy

measurement system. These are termed the "biased estimating system," "the operation to remove bias" and the "output meter." The linear noisy measurement system is restricted to those systems for which (1) non-random errors can be neglected, (2) the system elements of the biased estimating system are linear and stable and (3) the physical quantity to be measured is a constant applied at t equal zero. The biased estimating system produces an estimate of the quantity being measured. The output of the biased estimating system, however, must be displayed by a physical device in order that it can be read, and furthermore its output must be calibrated in terms of the physical quantity being measured. The output meter and operation to remove bias performs these two additional operations.

The nature of the biased estimating system makes it the component which operates first on the physical quantity to be measured. The operation to remove bias can either follow the biased estimating system or the output meter. The mathematical models for these components are used to investigate the two possible connections and the conclusion is reached that theoretically the two connections are essentially equivalent. However, several practical differences are discussed which make it seem desirable to place the operation to remove bias after the output meter. Since the two connections are substantially equivalent theoretically, the important aspects of the analysis and discussion applies to either.

Practical physical assumptions in conjunction with the mathematical model lead to the conclusion that the bias and mean

square errors due to the output meter are substantially independent of those for the biased estimating system. Thus these two operational components can be considered separately and their errors added to give the overall system error. If the operation to remove bias is performed on paper or by calibration of the output meter, its errors will be negligible relative to those of the other two components.

The mathematical model developed for the biased estimating system includes the effect of two types of noise, namely, that applied a long time before the signal and that applied with the signal. Both types of noise are assumed to be Gaussian random processes, but one type is a stationary process and the other is not. The model accounts for any number of distinct independent noise sources of these types. The other parameters of the biased estimating system are the impulse and step responses of its component element groups.

General expressions which apply in the transient as well as the steady-state condition are derived for the bias and mean square errors of the biased estimating system. The general expression for bias error due to the biased estimating system shows that this error can always be reduced to zero by the operation to remove bias.

The mean square error due to the biased estimating system is expressed in terms of the normalized step and impulse responses for the components of the biased estimating system. The normalization is performed with respect to the gain of the component element groups and with respect to an arbitrarily selected characteristic frequency. The expression for the mean square error in terms of

these normalized responses shows that the effect on mean square bias error of a given noise source is reduced in proportion to the reciprocal of the gain between the point of application of the noise and the system input. The same expression shows that if the product of the characteristic frequency and the time at which a reading is taken is held constant, then mean square error due to the biased estimating system is proportional to the characteristic frequency.

The effect of several readings of the measurement system output on the mean square error due to the biased estimating system is investigated for a special case. The results show, among other things, that the mean square error of an estimate of the quantity being measured formed from two readings could be less than the arithmetic mean of the mean square errors for single readings taken in two independent measurements.

A significant lower bound has been established for the mean square error of the biased estimating system. This lower bound is

$$\frac{\pi}{t}(SA_1^2 + B_1^2)$$

where t is the time at which a single reading is taken, S is the true value of the constant being measured, and A_1^2 and B_1^2 are the power per unit bandwidth of the noise sources applied at the system input. Several tractable bounds for the mean square error of the biased estimating system in special cases are obtained.

The mathematical model developed for the output meter accounts

for two types of errors in this device. These errors are called movement error, a term which applies to all errors which cause the actual indicator position to differ from its "true" position, and round-off error which accounts for the discrete nature of the scale divisions. The parameters of the output meter are its sensitivity, its smallest detectable scale division, γ , and the mean square value of its movement error, σ_m . The mean value of movement error is assumed to be zero.

Since the quantity to be measured is unknown, it is pointed out that round-off error cannot be precisely determined. It can, however, be bounded, and analysis of the mathematical model for the output meter shows that round-off error is bounded between $\pm\gamma/2$.

The mathematical model also shows that bias error for the output meter is equal to round-off error so that the bounds given above are in fact the bounds on the bias error for the output meter. The mean square error due to the output meter is shown to lie between $\gamma^2/4 + \sigma_m^2$ and $\gamma^2/2 + \sigma_m^2$.

Analysis of the case of several readings of the measurement system output shows that the movement error contribution to an estimate formed from n reading of the system output is $1/n^{\text{th}}$ that for a single reading. For the same case round-off error is reduced but by a lesser amount.

The study is concluded with the discussion of a practical design problem which illustrates the concepts and the use of some of the general results of this work.

CHAPTER I

INTRODUCTION

During the past ten years many discussions of the applications of probability and statistical theory to engineering problems have appeared in the literature. Representative of this work are the books by Middleton (1)*, and Freeman (2) and the series of papers by Rice (3). Application of these methods, however, to a theoretical study of the principles of measurement systems has been neglected. Although some excellent papers have appeared that discuss the application of statistical methods to specific measurement problems, for example the papers by Blackman and Tukey (4), Davenport and Middleton (5), and Spetner (6), very little work has been presented that treats measurement systems with deliberate generality.

The purpose of this study is to begin the task of formulating a precise mathematical theory of measurement systems. Because this work enters a field that is not well established it would be difficult to isolate a specific problem which can be investigated in exhaustive detail. Instead an attempt has been made to isolate a non-trivial class of measurement systems and develop a complete mathematical model for this class of systems. It has become apparent that even for the restrictive class of systems chosen the existence of a precise mathematical model has uncovered more specific detailed

*Numbers refer to bibliography.

questions than could be answered by a research program of reasonable length. Therefore the emphasis in this investigation has been on the overall formulation of the problem. This has necessitated leaving to later work some of the problems which are of specific interest but which are of secondary importance to the general development. For example, Chapter VII discusses the problem of using several readings of the measurement system to form an estimate of the quantity being measured. As is pointed out in that chapter the present discussion leaves unsolved the interesting, but specific, problem of choosing the best functional form for the estimate and establishing a procedure for determining all the pertinent parameters so as to minimize the mean square error of the estimate. There are other specific problems in this category.

With reference to what has been done, the first part of this study is devoted to identifying the logical operations in a measurement system, and then developing a detailed mathematical model for each logical part of the system. Since noise is an important factor in the class of systems being considered it has been necessary to discuss the nature of errors in noisy measurement systems in this part of the work.

The final portion of the study brings together the mathematical models for the components of the measurement system and discusses the measurement system as a whole both theoretically (in Chapter VII) and from a less mathematical and more practical point of view (in Chapter IX). The final chapter is an example which is worked out in detail to illustrate the ideas developed in the study.

CHAPTER II

FORMULATION OF A GENERAL LOGICAL STRUCTURE FOR A CLASS OF MEASUREMENT SYSTEMS

A logical operational structure for a class of measurement systems can be based on five distinct operations performed by a transducer, subtracting device, amplifier, reference source, and output meter related in the manner shown in Figure 1. These logical operations are identified from the equations describing a physical system and thus each distinct logical operation is not necessarily performed by a distinct physical component.

Each of these logical operations will now be discussed. In the usual measurement system the quantity to be measured cannot be used directly to deflect a meter, hence it is necessary to use a transducer whose output (or amplified output) can do so. Such a transducer operates according to some physical law in such a way that the transducer output is a variable which can be worked with conveniently and which is functionally related to the magnitude of the input. For example, if an electrically deflected meter is to be used to measure pressure it is necessary to use a transducer to obtain a voltage or current proportional to pressure. As a rule, if the type of meter is specified, the choice of transducer for use in measuring a specified physical quantity is limited by the number of physical devices which convert from one variable to the other.

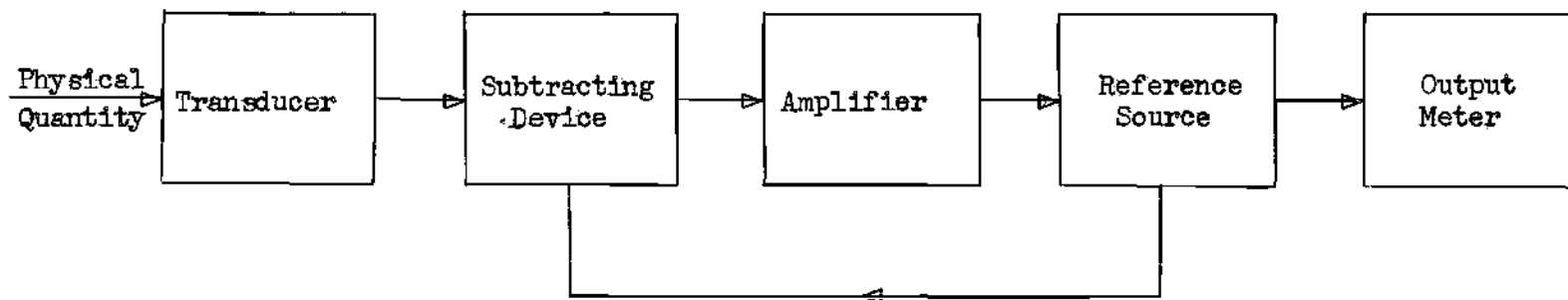


Figure 1 Logical Block Diagram for a Class of Measurement Systems

A transducer is usually inherently frequency selective, and furthermore the output of every physical transducer contains not only the signal or output quantity directly related to the physical quantity to be measured, but also some noise or extraneous output originating in the transducer or its environment.

Null type instruments are designed in such a way that the transducer output is measured by a closed loop system. That is to say, the transducer output is compared to the magnitude of a calibrated reference and the difference in these quantities is used to adjust the calibrated referenced until the difference approaches zero. The component labeled subtracting device is assumed to perform the operation of comparing the transducer output to the magnitude of the calibrated reference by subtracting one variable from the other.* This component unavoidably introduces both noise and inertia.

The output of the subtracting device is not in general sufficient to actuate the reference source, hence an amplifier is necessary. The primary function of the amplifier is to provide necessary gain, but the amplifier unavoidably introduces in addition to the gain both noise and inertia. The inertia causes the amplifier to be frequency selective. In some applications additional frequency selectivity is intentionally added in this element to improve the overall system behavior.

*There are no doubt measurement systems for which the physical quantity to be measured is compared directly to a reference physical quantity. In this case the subtracting device would precede the transducer. Although some details of the analysis would be different, such measurement systems would fit the structure developed above with minor modifications.

The reference source represents the calibrated standard necessary in any null or bridge type instrument.* The main purpose of this component is to provide at its feedback output a variable standard quantity of the type present at the transducer output. This standard is adjusted in response to the amplifier output by a motor or human operator until its feedback output is equal to the transducer output, at which time the output of the subtracting device would become zero except for the noise. For non-bridge type systems this feedback output would be zero.

The second output of the reference source goes to the output meter. The details of this connection to the output meter depend on the type of measurement system. Thus at least two distinct cases must be considered to account for bridge or null type systems and non-bridge type systems.

In all cases the reference source will introduce both noise and inertia into the system.

The primary purpose of the output meter is to provide an indication, on a scale or other display, of a real number proportional to the instantaneous magnitude of the appropriate output of the reference source. In most cases the output meter is calibrated so that its output is in the units of the quantity to be measured, however, if the output meter is not so calibrated the calibration must be performed after a reading is made. Thus this calibration is a necessary operation in any measurement system and it will be assumed to be included in the

* Some null type instruments such as lobe comparison radar compare two observations with each other rather than with a standard. Such a system could be fitted into the representation described although the details will not be discussed here.

output meter of the logical diagram of Figure 1. In later work the calibration operation will be made more general and identified as an operation distinct from the output meter.

The output meter introduces errors of several types, the most important of which arises because a reading of the output display is limited to some fixed number of significant figures. The output meter also introduces both noise and inertia.

CHAPTER III

GENERAL DISCUSSION OF THE NATURE OF NOISY MEASUREMENT SYSTEM ERRORS

The discussion of Chapter II indicated that every element of a measurement system will potentially introduce noise so that the output of each element will in general differ from what it should be under ideal noise-free operation. In the mathematical models which will be developed below, system noise will be taken into account so that each element output must be described mathematically as a stochastic process. It is the purpose of this chapter to make several comments relative to the nature of errors in noisy systems and how they will be treated mathematically. It is definitely not the purpose of this chapter to discuss either the philosophy of applying statistical techniques to physical systems or the details of mathematical probability theory, although these topics are relevant. Both of these topics represent well defined areas in themselves and have been treated in detail elsewhere.*

A complete measurement system as well as a single measurement system element can be represented by the diagram of Figure 2. All of the parameters of the input, $x(t)$, but one will be assumed to be known.

* See, for example, Brillouin (7) for a discussion of the philosophy of applying statistical theory to physical systems, and Cramér (8) for a treatment of mathematical probability theory.

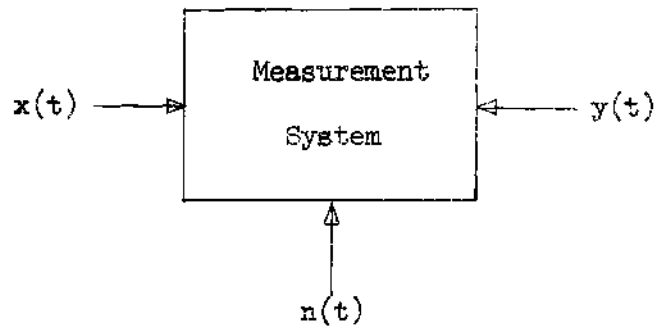


Figure 2 A Measurement System

For example, if the diagram represents a complete measurement system, $x(t)$ will be the quantity being measured and the purpose of the system will be to form an estimate of an unknown parameter, S , of $x(t)$. In other cases $x(t)$ may be either a known function of time with one unknown parameter, or a stochastic process with all but one statistical parameter known or assumed to be known. The point of view to be adopted here will be that all of the statistical parameters of the noise processes (represented in Figure 2 by $n(t)$) are known. In practice the parameters of sample functions from a noise process can be estimated, but it is not possible to measure the statistical parameters of the complete process. Thus the statement above represents an assumption which could be based on a knowledge of the nature of the mechanisms producing the noise.

The procedure to be followed in this work will be to use the assumed knowledge of the processes $x(t)$ and $n(t)$ to compute the statistical parameters of the complete process $y(t)$ in terms of the unknown parameter. Although the statistical parameters of the complete process $y(t)$ cannot be measured, they can be used to express the statistical properties of practical sample functions, which are regarded as random variables prior to making the measurement. Thus probability statements can be made concerning the result of physical samples of the $y(t)$ process. For example, if the complete $y(t)$ process has a mean, m , and a standard deviation, σ , then the inequality*

* This inequality is due to Bienaymé and Tchebycheff, see for example Cramér (8).

$$P \left[\left| y(t_1) - m \right| \geq C\sigma \right] \leq \frac{1}{C^2} \quad (1)$$

where $P \left[\right]$ indicates the probability of an event and C is a positive constant, gives a bound on the probability that the value obtained in a single reading of $y(t)$ will differ from the mean of $y(t)$ by more than $C\sigma$. This expression is evaluated and plotted for various values of C in Figure 3. The probability that $y(t_1)$ differs in magnitude from the mean of $y(t)$ by more than $C\sigma$ is also plotted in Figure 3 for the special case that $y(t)$ has a Gaussian distribution function. The curves are plotted in such a way that the ordinate gives the minimum probability that $\left| y(t_1) - m \right|$ is less than the corresponding abscissa.

The remarks above relative to the procedure to be used in considering noisy measurement systems may also be made from a slightly different point of view. All of the measurement system inputs, including noise, will be assumed to be sample functions from random populations which are assumed to be specified by statistical parameters which are known with the exception of the single parameter being measured. The known properties of the measurement system will then be used to compute the nature of the measurement system output space. That is to say the underlying assumptions will make it possible to describe, within an unknown parameter, the space of all possible measurement system outputs. In any given experiment with the measurement system particular sample functions from the noise processes will be applied and a single measurement or perhaps several measurements of the system output will be made. This particular system output

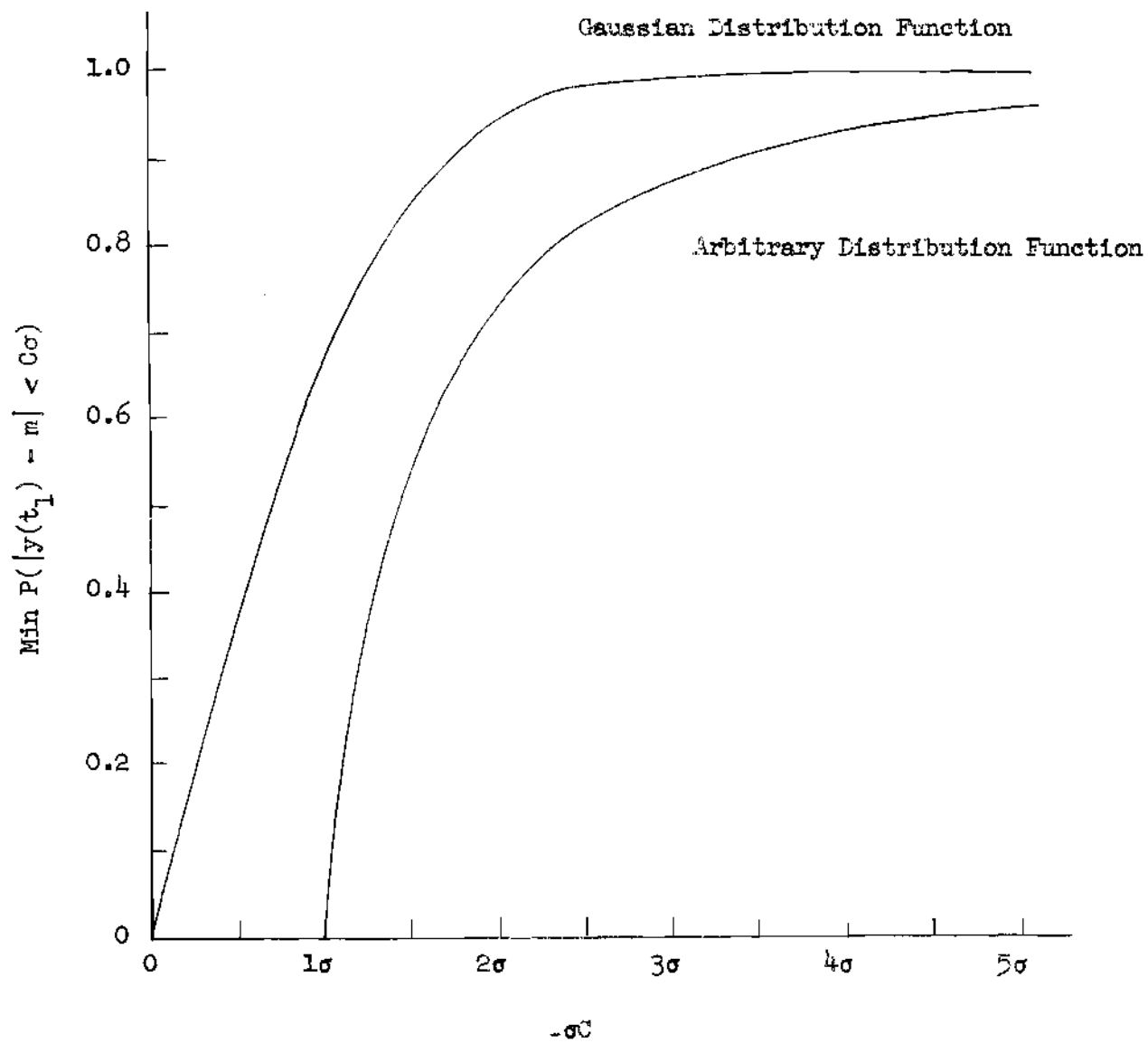


Figure 3 A Plot of the Minimum Value of

$$P[|y(t_1) - m| < c\sigma] \text{ vs } c\sigma$$

or some function of the several outputs will then be used as an estimate of the true measurement system input parameter which is being measured. Since the quantity being measured is unknown, the quality of the estimate can only be judged in the probability sense. That is the knowledge of the space of all possible outputs can be used to make statements such as the inequality of relation (1) giving the probability of a specific measurement system output being in a particular range about the quantity which it estimates.

As indicated above, prior to making the measurement all of the statistical parameters of the output but one will be assumed to be known. Thus in particular its probability distribution function will be known except for a single parameter. Conceptually the statistical moments of the measurement system output could be calculated using this distribution function. These moments would thus also depend on the unknown parameter. In this work the moments will be calculated by averaging the appropriate function over the ensemble of values that result from the use of all possible sample functions from the noise process. Such an average or expected value of a function, say $y(t)$, could be denoted $E_S y(t)$ to indicate that the average depends on the unknown parameter S . The subscript S will be implied but not explicitly shown in the remainder of this work.

The measurement system problem is essentially one of designing a physical system which can be used to produce an estimate of the quantity to be measured. Two important general types of error could be designated "bias error" and "mean square error." Bias error, which is taken from the statistical use of the term bias, is defined for a random variable $y(t)$ as

$$\text{bias error} = E y(t) - S \quad (2)$$

where S is the parameter to be measured. Mean square error is defined as

$$\text{mean square error} = E (y(t) - S)^2 . \quad (3)$$

These two errors are important because of their relation to the inequality of relation (1). This inequality, or a stronger one that could apply in a specific case, gives a measure of the probability that a single reading, $y(t_1)$, differs in absolute value from its mean, m , by more than some multiple of the standard deviation of $y(t)$, σ . The importance of bias and mean square error lies in the fact that bias error is a measure of the difference between m and S , and the fact that for small bias error, mean square error is approximately equal to σ^2 . For example, if the bias error were equal to zero, the mean square error would be equal to σ^2 so that mean square error could be used directly in relation (1) to determine with a given probability the largest possible difference between a single reading of a measurement system and the true value of the quantity being measured. In general the bias error will not be zero so that mean square error is only approximately equal to the variance of $y(t)$. As a matter of fact there are cases of importance which arise in this study where the mean square error is itself a function of the unknown parameter S . In such cases the estimate of S given by the measurement system could be used to give an approximation to the mean square error. However, these uncertainties in the mean square error and in using the mean square error to approximate the standard deviation of

$y(t)$ do no more than widen the bounds than can be placed on $|y(t_1) - S|$ with a given probability.

In cases where the bias and mean square errors are independent of the unknown parameter, these two errors give a good measure of the performance of a measurement system. On the other hand if either bias or mean square error depends on the unknown parameter, S , then these errors are not suited to consider, for example, questions relating to the optimum value of S to use in making a measurement. In such cases fractional bias error and fraction mean square error defined as

$$\text{fractional bias error} = \frac{E y(t) - S}{S} \quad (4)$$

$$\text{fractional mean square error} = \frac{E(y(t) - S)^2}{S^2} \quad (5)$$

seem more appropriate.

The remarks above have indicated that bias and mean square errors are good indications of measurement system performance. This statement is particularly true relative to the restricted class of systems which will be treated in the major part of this study. It is clear, however, that these two errors are not the only errors and that these two numbers, or any two numbers, are not adequate to completely describe a non-trivial process such as a measurement system.

In spite of this latter comment bias and mean square errors (along with the fractional errors) will be the criterion of measurement system performance used in this study.

CHAPTER IV

FORMULATION OF A MATHEMATICAL MODEL FOR THE TRANSDUCER, SUBTRACTING DEVICE, AMPLIFIER AND REFERENCE SOURCE

Chapter II outlines a logical structure into which a large class of measurement systems will fit. In this chapter a mathematical model suitable for describing the transducer, subtracting device, amplifier, and reference source will be developed. The mathematical model, developed in this chapter specifically for measurement systems fitting the logical structure of Chapter II, will apply to a much wider class including systems other than measurement systems. Although the primary emphasis in this study will be on systems which fit the logical structure of Chapter II, it seems reasonable to maintain as much generality as possible in this chapter and apply restrictions only if they reduce the complexity of the mathematical model.

To develop a tractable mathematical model a number of assumptions are necessary. These assumptions fall into three classes. The first class of assumptions applies specifically to the use of the mathematical model in describing measurement systems. The other two classes of assumptions have to do with the nature of the noise introduced by the circuit elements and the nature of the circuit elements themselves.

The first class of assumptions defines the type of physical system and the emphasis chosen in this investigation. This investigation will be restricted to that type of measurement system designed to measure a constant physical quantity. The emphasis will be on random errors and therefore it will be assumed that non-random errors are negligible.

Each of these assumptions is of course restrictive, but the resulting class of instruments is important enough to warrant consideration.

The second class of assumptions has to do with the nature of the noise introduced by the elements described in Chapter II. Reference to the literature reveals that there are at least eight distinct physical mechanisms which could potentially produce this type of noise. The type of noise considered will be restricted to that which adds to the signal, since this is felt to be the most important type in most physical measurement system. Such noise sources have been the subject of much recent work which has resulted in a number of books and papers reviewing the subject. Representative of this work are the books by Van der Ziel (9) and Smullin (10) and the paper by Jones (11). There seems to be some disagreement in terminology and in the method of isolating the various physical mechanisms responsible for producing noise, but there is agreement in the mathematical description of the predominant types of physical noise. In the literature surveyed most common types of physically produced noise are described mathematically as stationary Gaussian stochastic processes. Such processes are completely specified by the mean and either the correlation function or its Fourier transform the power spectral density function provided both functions exist.* Two power spectral density functions are in widespread use to approximate the common sources of noise. One of these is assumed to be a constant for all frequencies in the range of interest and zero elsewhere, and the other varies with frequency as $1/\omega^\alpha$ where ω is radian frequency and α is near unity. The former choice for the power spectral density function is used to

*The functions and relations referred to here are defined and discussed for example by Laning and Battin (12).

approximate the characteristics of Johnson noise, shot noise and radiation noise, three important sources of noise in measurement systems. The correlation function corresponding to this power spectral density has the form $\sin \omega_1 \tau / \tau$ where ω_1 is the highest noise frequency and τ is the delay.

For this work it has been decided to choose the mathematical description of the noise so that it will approximate specifically Johnson noise, shot noise, and radiation noise. Although the single representation given above is in common use in the literature for these three types of noise, it will be necessary to use two different descriptions in this investigation where transient effects are to be taken into account. Since a mathematical description of shot noise and radiation noise adequate for a discussion of system transients has not been found in the literature it will be necessary to modify an existing treatment to obtain an adequate mathematical description.

As mentioned above the noise processes under consideration are Gaussian stochastic processes in which time is a parameter. The nature of such processes is specified by the value of the two integrals which define the mean and correlation function for the process. The mean and correlation function can be defined as time averages or as averages over an ensemble which is physically interpreted as a number of identical systems. If the process is stationary and ergodic* the averages computed in the two alternative ways have the same value. In the discussions of noise found in the literature the noise process has been assumed to be stationary and time averages have been chosen to define the mean and correlation function.

* Definitions and a discussion of the properties referred to here can be found for example in Laning and Battin (12).

As mentioned above the effect of the three types of noise being considered is often taken into account by introducing random noise sources whose outputs have zero mean and a correlation function, $\phi(t_1, t_2)$, given by

$$\phi(t_1, t_2) = E_n^2 \frac{\sin \omega_1(t_2 - t_1)}{t_2 - t_1} \quad (6)$$

where ω_1 is the highest noise frequency and E_n is the rms value of the noise.

If system transients are to be considered it is not reasonable to assume that either the shot noise process or the radiation noise process is stationary, and hence for these cases the time and ensemble averages are not equal. The natural choice for defining averages in the non-stationary case is over the ensemble. Thus from an intuitive point of view it seems reasonable to choose for shot noise and radiation noise essentially the same expressions for the mean and correlation functions as commonly defined by time averages, but define these averages over an ensemble. From the ensemble point of view, however, the fact that the rms value of the noise may change slowly with time can be taken into account by allowing E_n in equation (6) to be a function of time. Using the symbols $\phi'(x)$ and E_n' for the new case, equation (6) becomes

$$\phi'(t_1, t_2) = E_n'(t_1)^2 \frac{\sin \omega_1(t_2 - t_1)}{t_2 - t_1} \quad (7)$$

This type of correlation function and a mean of zero will be assumed for the output of the generators used to approximate the effect of shot noise and radiation noise. Before proceeding, however, several additional

comments justifying this choice for the correlation function specifically for shot noise will be made. In this connection reference will be made to a paper by Rice (13) as a standard discussion of shot noise.

First consider a noise-free system of the type which may represent a measurement system. At all points in this system the voltages and currents representing the signal will vary with time in the transient state after a signal is applied to the input. Thus at a general point in the system the signal, $s(t)$, will vary with time even if the applied input is a constant. When noise is applied to the system it is reasonable to assume that the ensemble average of signal plus noise is equal to signal, $s(t)$.

Now turn to Rice's derivation of the parameters of shot noise. Most of the physical discussion lies in the mathematical definition of the probability density function, $p(x)$, for the output, $I(t)$ of the noisy device. Shot noise is assumed to arise because of random fluctuations in the electron stream which produces $I(t)$. In his discussion Rice (13) assumes that the probability of an electron arriving at the anode in the time $t, t + \Delta t$ is $\bar{n}\Delta t$ where \bar{n} is the average number of electrons arriving per second. He computes \bar{n} by counting the number of electrons, K_1 , in each of a number, M , of successive intervals of time length T using the equation

$$\bar{n} = \lim_{M \rightarrow \infty} \frac{1}{MT} \sum_{i=1}^M K_1 .$$

From this starting point he obtains for $p(x)$ the expression*

* Rice later shows that this probability density approaches a Gaussian one as $\bar{n} \rightarrow \infty$, a condition which is closely approximated in all cases of interest.

$$p(x) = \frac{(\bar{n}T)^x}{x!} e^{-\bar{n}T}.$$

It seems reasonable to modify Rice's work by defining \bar{n} so that this average is computed by using samples taken from different members of an ensemble rather than samples taken sequentially in time. This change in point of view allows \bar{n} to be defined as a function of time proportional to the signal, $s(t)$, or in a more general case proportional to $Ee(t)$ if $e(t)$ is signal plus noise. Thus as long as $s(t)$ varies slowly relative to the time, T , required to obtain a large number of electron arrivals at the anode, it would seem reasonable to modify the expression for $p(x)$ to $p'(x)$ given by

$$p'(x) = \frac{(As(t)T)^x}{x!} e^{-As(t)T}$$

where A is a constant. This modification of $p(x)$ does not affect the other details of Rice's work so that his final expressions can be modified by replacing \bar{n} by $A|Ee(t)|$. Rice's result of primary interest is the expression for the correlation function of the shot noise given by him as equation 2.6-2 (13). This equation gives the correlation function of the output of a linear filter with shot noise as its input. For the purposes here Rice's equation is further modified to give the correlation function of primitive shot noise (i.e., shot noise before the linear filter) with a zero mean. Of course since the process is non-stationary power spectral density is no longer a useful term and Rice's results in this connection are not applicable.

Thus in the measurement system model being developed the effect of shot noise and radiation noise will be taken into account by appropriate noise sources each producing an output, $N_1(t)$, which is a

Gaussian stochastic process having zero mean and a correlation function $\Phi_1(t_1, t_2)$ given by

$$\Phi_1(t_1, t_2) = A_1^2 |Ee_1(t_1)| \frac{\sin \omega_1(t_2 - t_1)}{t_2 - t_1}$$

where A_1 is a constant, $|Ee_1(t_1)|$ is the magnitude of the ensemble average of signal plus noise at the point of generation of the noise, and ω_1 will be taken to be very large.

Unlike shot noise and radiation noise the mechanism producing Johnson noise is such that Johnson noise is present continuously and hence this type of noise can be represented by a stationary stochastic process. As is well known from the literature, Johnson noise can be represented by a stationary Gaussian stochastic process with mean zero and power spectral density which is uniform to very high frequencies. Thus the generator type chosen to approximate Johnson noise will have an output $n_1(t)$ with mean zero and correlation function, $\psi_1(t_1, t_2)$ given by

$$\psi_1(t_1, t_2) = B_1^2 \frac{\sin \omega_2(t_2 - t_1)}{(t_2 - t_1)}$$

where B_1 is a constant and ω_2 will be assumed very large.

In cases where several noise generators are applied to a system the assumption will be made that there is no correlation between noise generator outputs; furthermore zero correlation will be assumed between signal and noise.

The third and final class of assumptions necessary to formulate

a tractable mathematical model has to do with the nature of the system elements. It will be assumed that the behavior of each element can be described by a linear equation, and that all elements are physically realizable. Furthermore it will be assumed that all system elements have transient responses which decay with time.

A mathematical model will now be developed subject to the second and third classes of assumptions listed above. In later paragraphs of this chapter application of this mathematical model to systems fitting the logical structure of Chapter II will be made.

The assumptions listed above restrict the differential equations describing the class of systems being considered to that class of equations which describe linear, active, stable, electrical networks. This being the case such systems may be analyzed by the well known techniques for analyzing electrical networks.

The particular choice of form for the mathematical model is shown schematically in Figure 4. This form has been chosen because of the simple way in which the noise sources enter the model. The general applicability of the model will be demonstrated below and its utility in later chapters.

Definition of Symbols in Figure 4:

$S(s)$ - input

$\bar{N}_1(s)$ - Frequency domain representation of the output of the 1th noise source representing the effect of shot noise and radiation noise.*

* $\bar{N}_1(s)$ and $\bar{n}_1(s)$ are to be interpreted as the LaPlace transforms of particular sample functions from the $N_1(t)$ and $n_1(t)$ ensembles. The frequency domain representation for the system elements of Figure 4 have been chosen for convenience in dealing with the non-random aspects of the system. In later work dealing with random aspects of the system a time domain representation will be chosen.

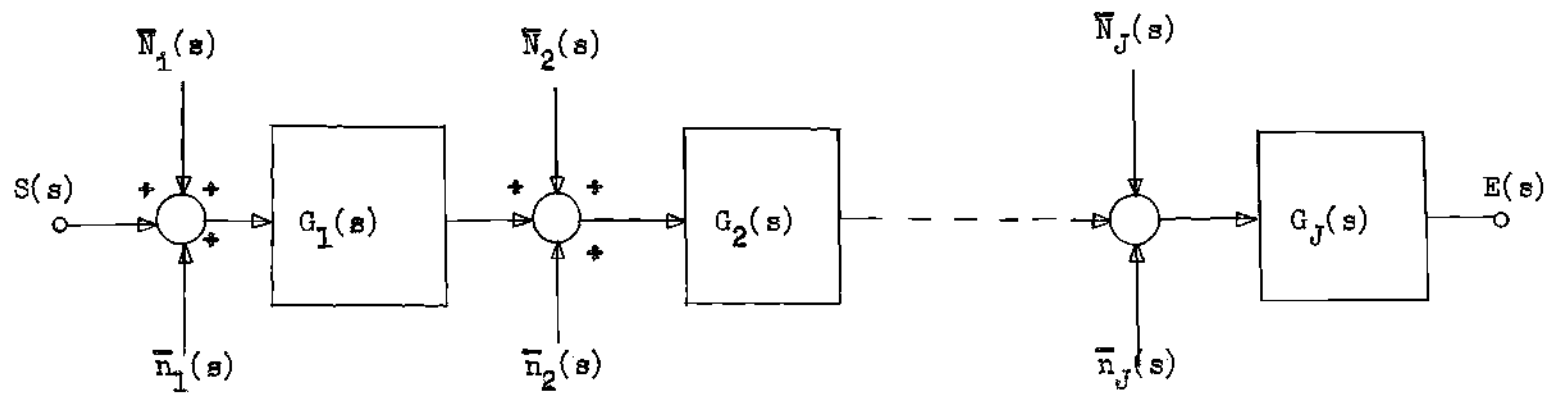


Figure 4 Schematic Representation of Mathematical Model

$\bar{n}_i(s)$ - Frequency domain representation of the output of the i^{th} noise source representing the effect of Johnson noise.

$G_i(s)$ - The transfer function between the points of application of the i^{th} and $(i+1)^{\text{th}}$ sources in Figure 4. It should be noted that $G_i(s)$ in Figure 4 is in general a function of more than one transfer function in the physical system.

$E(s)$ - output.

The following argument may be used to show that all systems of the type being considered may be represented by the diagram of Figure 4. For the class of systems being considered superposition applies. Hence for a system with an input $S(s)$, and J noise sources $\bar{n}_1(s)$ and $\bar{N}_1(s)$ the output $E(s)$ may be expressed as

$$E(s) = \sum_{i=1}^J T_i(s) [\bar{n}_i(s) + \bar{N}_i(s)] + T_1(s)S(s) \quad (8)$$

where each of the $T_i(s)$ is the transfer function between $E(s)$ and the particular source multiplying the $T_i(s)$, and the initial conditions have been assumed to be zero.** The equation for the network of Figure 4 is

$$E(s) = \sum_{i=1}^J \left[[\bar{n}_i(s) + \bar{N}_i(s)] \prod_{j=1}^J G_j(s) \right] + S(s) \prod_{j=1}^J G_j(s) \quad (9)$$

From equations (8) and (9) it is clear that the $G_i(s)$ can be expressed in terms of the $T_i(s)$ by solution of the J equations

** The effect on the output of non zero initial conditions could be taken into account by appropriate modification of the input $S(s)$ since the network is linear. Since the point of application of $S(s)$ retains its identity in the network described by equation (9) assuming that initial conditions are zero does not restrict the present argument.

$$T_1(s) = \prod_{j=1}^J G_j(s), \quad 1 = 1, 2, \dots, J. \quad (10)$$

Thus since equation (9) follows from equation (8) the network of Figure 4 to which equation (9) applies is a general representation.

It should be noted that for a given physical system the representation of Figure 4 is not unique until the noise sources are numbered. Thus different numbering of the noise sources would result in different but equivalent representations. In the representation of Figure 4 dependent variables such as the output of the box having the transfer function $G_1(s)$ do not necessarily have physical significance. The only variables which are assured of having physical significance are the input, the output, and each of the noise source outputs.

In this connection it is interesting to establish sufficient conditions under which a dependent variable in Figure 4 other than the output will have physical significance. Presumably in analyzing a physical system there would exist a set of equations in which each variable has physical significance, such as for example the node or loop equations of an electrical network. This set of equations will be called the "physical" equations.

The condition that a variable in the physical equations be present also in the representation of Figure 4 is equivalent to the condition that equation (8) can be written as the simultaneous equations

$$\begin{aligned} E^*(s) &= \sum_{i \in \alpha} A_i(s) [\bar{n}_i(s) + \bar{N}_i(s)] + A_1(s) S(s) \\ E(s) &= \sum_{i \in \theta} B_i(s) [\bar{n}_i(s) + \bar{N}_i(s)] + B_1(s) E^*(s) \end{aligned} \quad (11)$$

where $E^*(s)$ is a variable in the physical equations, A_1 and B_1 are particular transfer functions, and α and θ are sets of integers such that $\alpha + \theta = J$. The above statement is true since each of the two equations above can be represented in the manner of Figure 4. The tandem connection of the representation of each of the equations would itself be in the form of Figure 4 and the variable $E^*(s)$, being the output of the first portion of the tandem connection, would thus appear as a variable in the new representation.

Sufficient conditions such that an equation of the form of equation (8) may be written as the simultaneous equations (11) will now be established. Any fixed system can be described by an equation of the form of equation (8) as

$$E(s) = \sum_{i=1}^J T_1(s) [\bar{n}_1(s) + \bar{N}_1(s)] + T_1(s)S(s) \quad (12)$$

where the $T_1(s)$, \bar{n}_1 , and \bar{N}_1 are appropriate to the given system. If a dependent variable $E^*(s)$ in the physical equations is given, it may be expressed in terms of the system parameters and independent variables as

$$E(s)^* = \sum_{i \in \alpha} A_1(s) [\bar{n}_1(s) + \bar{N}_1(s)] + A_1(s)S(s) \quad (13)$$

where the A_1 are transfer functions determined by the choice of $E^*(s)$ and α is a set of integers chosen so that the sum indicated on the right hand side of the above equation will extend over all noise sources affecting $E^*(s)$. Equations (12) and (13) may be combined to give for $E(s)$ the equation

$$\begin{aligned}
E(s) = \sum_{i \in \Omega} \left[T_i(s) - T_i/A_i A_i(s) \right] \left[\bar{n}_i(s) - \bar{N}_i(s) \right] \\
+ \sum_{i \in \theta} T_i(s) \left[\bar{n}_i(s) + \bar{N}_i(s) \right] + T_i/A_i E^*(s) .
\end{aligned}
\tag{14}$$

Now subject to the condition

$$\frac{T_i(s)}{T_i(s)} = \frac{A_i(s)}{A_i(s)} , \quad i \in \Omega
\tag{15}$$

equation (14) becomes

$$E(s) = \sum_{i \in \theta} T_i(s) \left[\bar{n}_i(s) + \bar{N}_i(s) \right] + \frac{T_i}{A_i} E^*(s) .
\tag{16}$$

Thus subject to the restraint expressed by equation (15), equation (12) may be written as the simultaneous equations (13) and (16). But since equation (12) is identical in form to equation (8) and equation (13) and (16) are identical in form to equations (11) it follows that equation (15) expresses a sufficient condition that equation (8) may be expressed in the form of equation (11).

Now consider application of the general representation of Figure 4 to measurement systems fitting the logical model of Chapter II and the first class of assumptions listed above. This class of measurement systems will be termed "linear noisy measurement systems". To conform with the first class of assumptions the input $S(s)$ is restricted to be constant. Such an input will be denoted S . Now since random errors have been restricted to those produced by the noise sources in Figure 4 and since non-random errors can be neglected the representation of Figure 4 can describe the transducer, subtracting device, amplifier and reference source of the logical diagram. These elements as a group form the heart

of the measurement system. The system is designed so that the output of the last of these four elements can be used to estimate the quantity being measured. In most cases direct use of this output, i.e. $e(t)$ in Figure 4, would yield a biased estimate of the quantity being measured. Thus the term "biased estimating system" will be applied to these four elements which are represented by the diagram of Figure 4. It should be noted that $e(t)$ exists only as a theoretical quantity and any practical determination of this output must involve an output meter.

The elements of the biased estimating system all belong to the same class, that is they can be described by linear equations, they are physically realizable, and they have transient responses which decay with time. Thus it will be convenient to treat these elements as a unit in later work.

The form of the logical diagram is such that the transducer output, which appears in the physical equations for the system, can in general be made to appear in the representation of Figure 4. This fact can easily be verified by examining the sufficient condition of equation (15) as applied to the logical diagram. Thus in many cases it will be expedient to distinguish two components of the biased estimating system, and to have the transducer output appear explicitly in the representation of Figure 4. In cases of this sort one portion of the representation of Figure 4 will give the behavior of the transducer, and the other portion will give the combined behavior of the subtracting device, amplifier, and reference source. In later work it will be convenient to designate the latter three elements as the "amplification system." With this notation the linear noisy measurement system may be represented by the diagram

of Figure 5. In this diagram L noise sources have been assigned to the transducer out of a total of J , $E_t(s)$ represents the transducer output, and other symbols have the same significance as above. Note that in the diagram of Figure 5 a generalized calibration operation called an "operation to remove bias" has been included as an operation distinct from the output meter. This separation of the calibration operation from the output meter will be continued in the remainder of the work.

The following example will illustrate the use of the general representation in analyzing a practical measurement system. Consider the diagram of Figure 6 as an equivalent circuit of a hot wire anemometer used to measure air velocity.* The general principle of operation is as follows. The value of the resistance R depends on its operating temperature. This resistance is placed in the stream of air whose velocity is to be measured. The moving air cools the resistor thus lowering its temperature and unbalancing the bridge circuit. By a feedback action the voltage E is increased with the result that the resistor current is increased so as to heat the resistor back to its original temperature. In the steady-state the bridge becomes rebalanced and the resistor current I is proportional to the air velocity, v , being measured.

Using the assumptions, $Z_1 \rightarrow \infty$, $Z_2 \rightarrow 0$, $R_1 \gg R_3$, $R_2 \gg R$, and $|A_1, A_2| \rightarrow \infty$ the following equations describing the system result:

$$R = R_0 - K_1(s)v + K_2(s)I^2$$

$$I = E/R_2 + e_n/R_2 - e_{n_2}/R_2$$

*This general type of instrument without noise is described for example by Partridge (14).

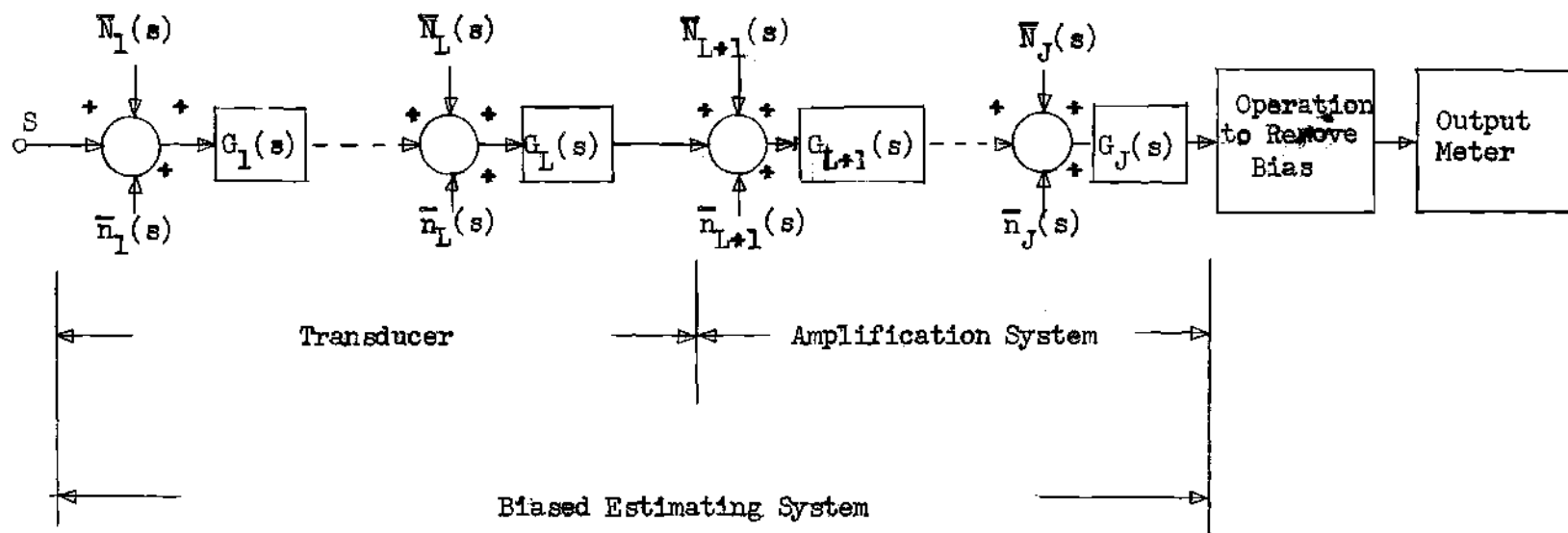


Figure 5 Schematic Representation of the Linear Noisy Measurement System

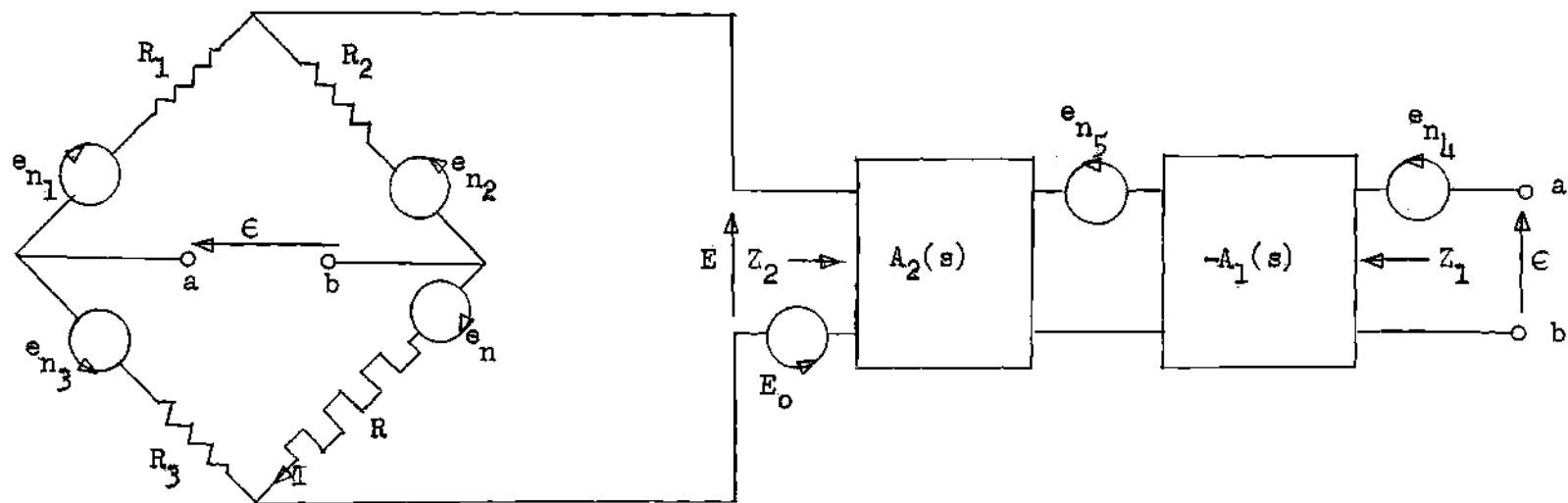


Figure 6 Equivalent Circuit of Hot Wire Anemometer

$$E = -A_1(s)A_2(s)\epsilon + E_o + A_2(s)e_{n_5} - A_1(s)A_2(s)e_{n_4}$$

$$\epsilon = e_n - e_{n_3} + R_3/R_1 E - R/R_2 E$$

These are the physical equations for the system. The argument s has been included in the right hand side of the above equations to call attention to those system components which are frequency dependent.

In problems of this type it is convenient to work with changes in the variables rather than the variables themselves. For small changes about the steady-state values increments may be approximated by differentials and products of differentials may be neglected. Since the noise voltages are considered to be small, changes in the noise voltage will be denoted by the same symbol as the noise voltage itself. Thus for small changes about the steady-state conditions the following equations approximate the system behavior

$$\Delta R = -K_1(s)\Delta v + 2K_2(s)E_o/R_2\Delta I$$

$$\Delta I = \Delta E/R_2 + e_n/R_2 - e_{n_2}/R_2$$

$$\Delta E = -A_1(s)A_2(s)\Delta\epsilon + A_2(s)e_{n_5} - A_1(s)A_2(s)e_{n_4}$$

$$\Delta\epsilon = e_n - e_{n_3} + R_3/R_1\Delta E - R/R_2\Delta E.$$

It is convenient to represent these equations by the flow graph of Figure 7a.* Examination of this flow graph or the equations above shows that

*See for example Mason (15) for a discussion of flow graphs.

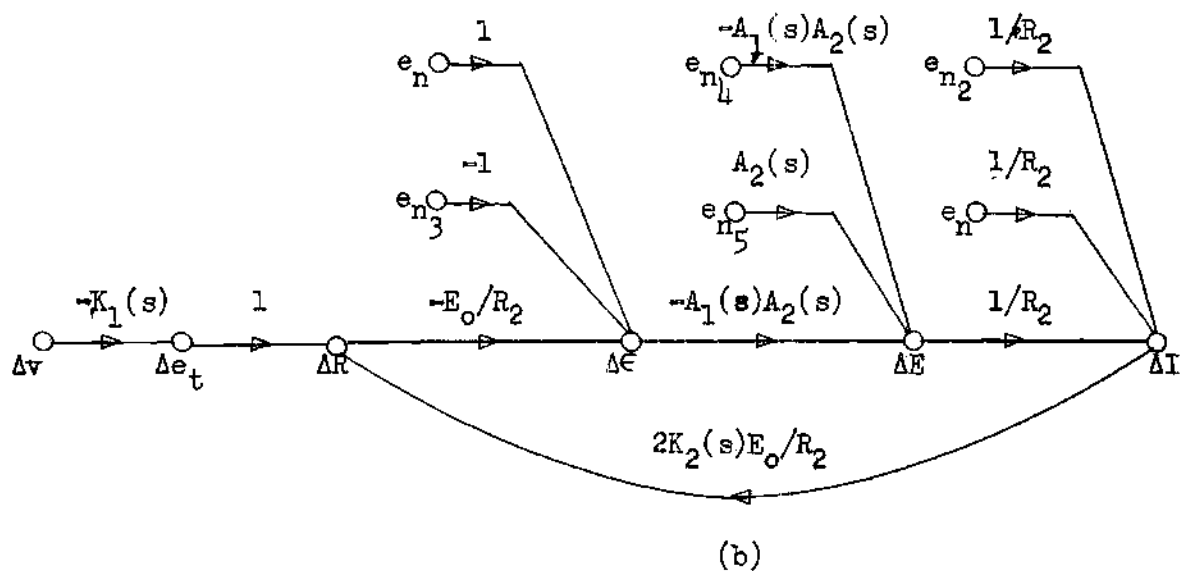
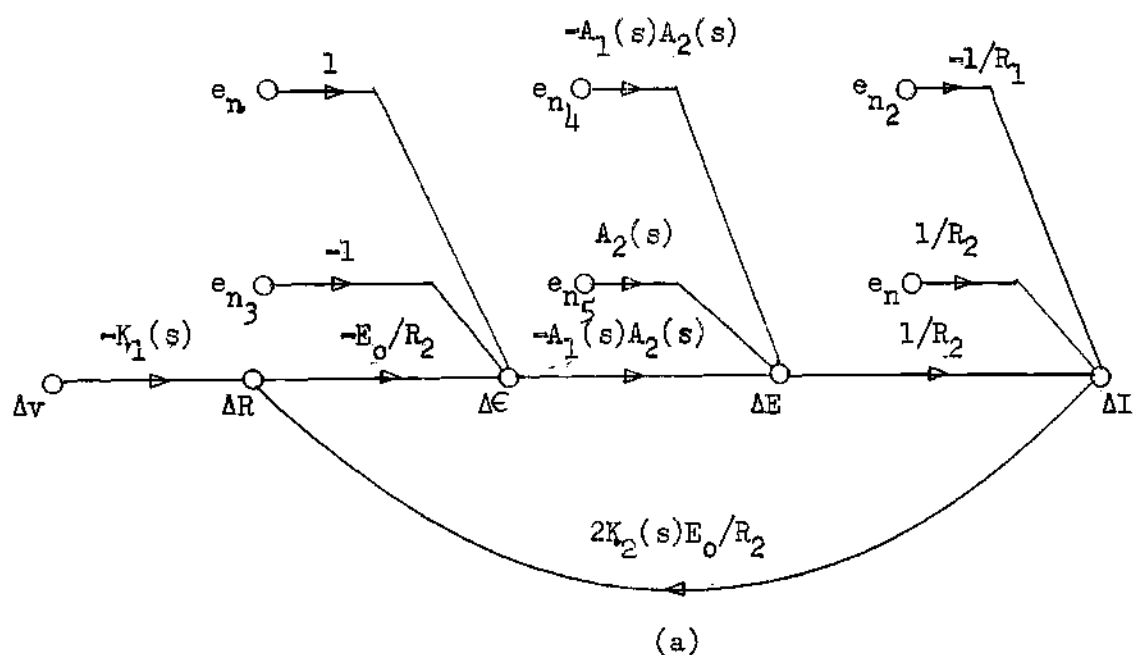


Figure 7 Flow Graphs of Hot Wire Anemometer

there is feedback from the output node, labeled ΔI , to the node labeled ΔR and that ΔR is the difference between the variables $K_1(s)\Delta v$ and $2K_2(s)E_o/R_2\Delta I$. Thus it is logical to identify ΔR as the output of the subtracting device of the general diagram of Figure 4. ΔR is a physical variable. In order to fit the operation of the physical anemometer to the general logical structure referred to above it is reasonable to introduce an additional node Δe_t in the flow graph of Figure 7a as shown in Figure 7b. This amounts to replacing the first equation of the set above by the two equivalent equations

$$\Delta R = \Delta e_t + 2K_2(s)E_o/R_2\Delta I$$

$$\Delta e_t = -K_1(s)\Delta v.$$

The variable Δe_t , which is not a physical variable, can then be identified as the output of the transducer.

The flow graph of Figure 7b can then be manipulated so as to remove all sources from inside the feedback loop. This results in the graph of Figure 8a. In this graph the sources have been re-labeled as follows:

$$n_1 = e_n$$

$$N_1 = 0$$

$$n_2 = e_{n_2}$$

$$N_2 = 0$$

$$n_3 = 0$$

$$N_3 = e_{n_4}$$

$$n_4 = 0$$

$$N_4 = e_{n_5}$$

$$n_5 = e_{n_3}$$

$$N_5 = 0.$$

Note that e_{n_1} , e_{n_2} and e_{n_3} are approximated as Johnson noise while e_{n_4} and e_{n_5} are approximated as shot noise.

The final steps in representing the hot wire anemometer in the manner of the general diagram of Figure 4 involve use of the flow graph rules to remove the feedback in Figure 8a and separate the points of application of the noise sources. This results in the final flow graph of Figure 8b which is in the form of the general diagram.

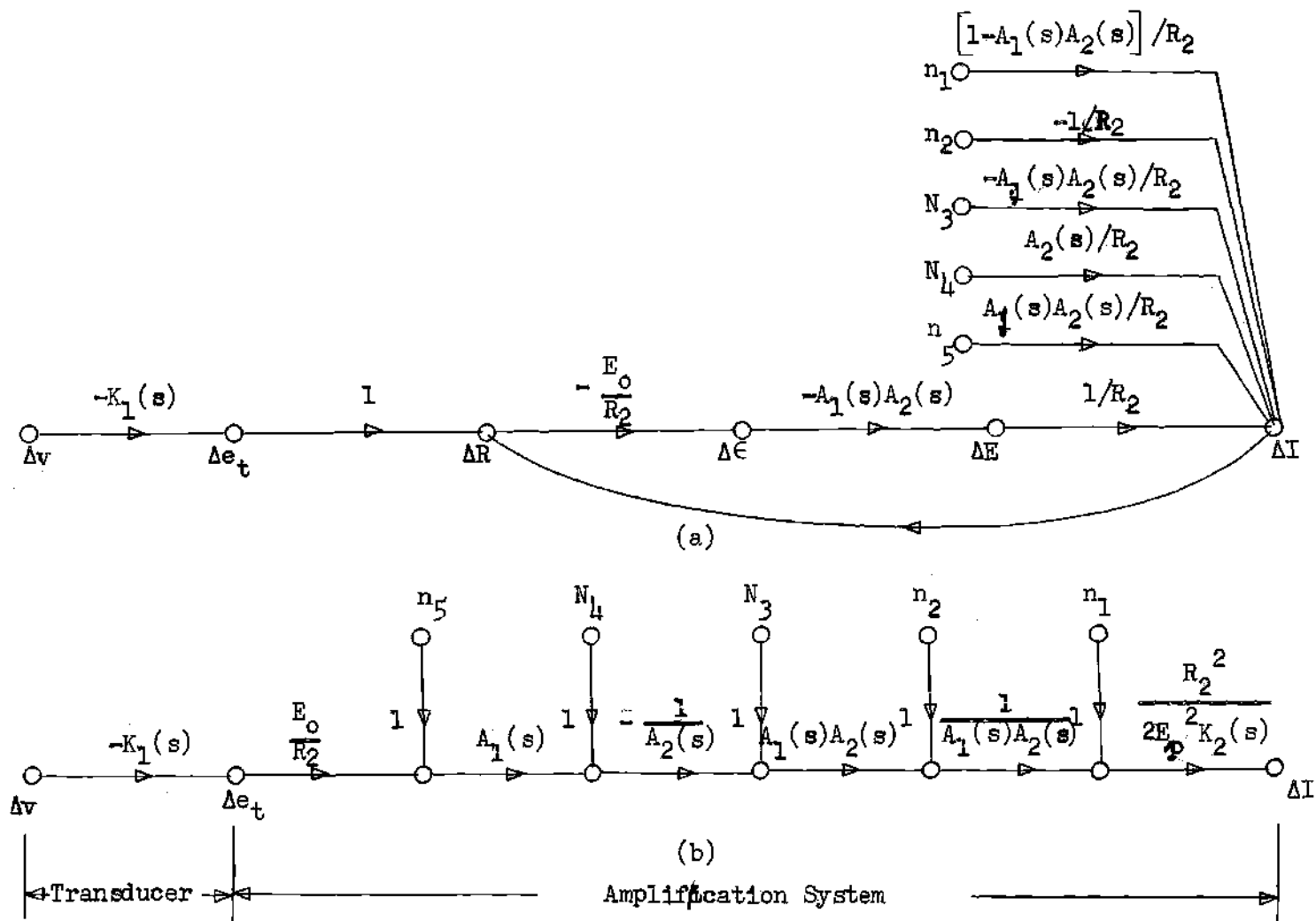


Figure 8 Flow Graphs for Hot Wire Anemometer

CHAPTER V

ANALYSIS OF THE BIASED ESTIMATING SYSTEM

In Chapter IV a mathematical model was established for the ~~trans-~~ducer, subtracting device, amplifier, and reference source of the logical system diagram. These elements as a group are termed the biased estimating system and are shown schematically in Figures 4 and 5 of Chapter IV. It is the purpose of this chapter to present a general analysis which will apply to the biased estimating system. The analysis will be made in such a way that the output of any intermediate element such as the transducer, for example, can be obtained by substitution of the appropriate transfer functions into the general result. The emphasis in this chapter will be on the mathematical details and the results of an analysis of the biased estimating system. Physical interpretation of these results and application to a general measurement system will be emphasized in a later chapter.

The analysis will apply to a group of elements such as shown in Figure 9. Before proceeding with the analysis the notation chosen for this part of the work will be presented, and the restrictions placed on the system elements will be summarized. The symbols have the following significance:

S - Constant applied to system at $t = 0$.

$N_1(t)$ - i^{th} noise source applied at $t = 0$.*

$n_1(t)$ - i^{th} noise source applied a "long time" before $t = 0$.

* i can have the values 1, 2, ---, J unless otherwise indicated.

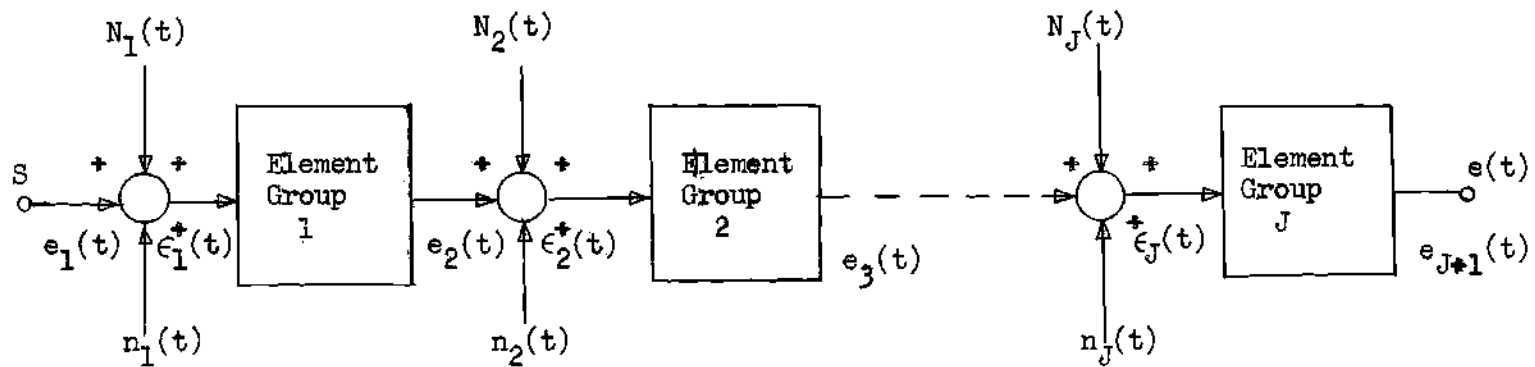


Figure 9 Representative Group of Elements from Biased Estimating System

$e_{i+1}(t)$ - Output of element group 1.

$e_1(t)$ - Input to element group 1.

$e(t)$ - Output of portion of biased estimating system being analyzed.

$\bar{e}(s)$ - $L[e(t)]$, similar notation for $L[e_1(t)]$,
 $L[e_{i+1}(t)]$, etc.

$\bar{g}_1(s)$ - Transfer function $\frac{\bar{e}_{i+1}(s)}{\bar{e}_1(s)}$, of i^{th} element group.

$g_1(t)$ - $L^{-1}[\bar{g}_1(s)]$ impulse response of i^{th} element group.

$\bar{h}_1(s)$ - Transfer function $\frac{\bar{e}(s)}{\bar{e}_1(s)}$.

$h_1(t)$ - $L^{-1}[\bar{h}_1(s)]$.

$\bar{k}_1(s)$ - Transfer function $\frac{\bar{e}_1(s)}{\bar{e}_1(s)}$, $1 = 2, 3, \dots, J$.

$k_1(t)$ - $L^{-1}[\bar{k}_1(s)]$, $1 = 2, 3, \dots, J$.

$G_1(t) = \int_0^t g_1(x)dx$ step response of i^{th} element group.

$H_1(t) = \int_0^t h_1(x)dx$.

$K_1(t) = \int_0^t k_1(x)dx$, $1 = 2, 3, \dots, J$.

$K_1(t) = 1$ all t .

$T(i,1) = \lim_{s \rightarrow 0} g_1(s)$, d-c gain of i^{th} element group.

$T(u;v) = \prod_{i=u}^v T(i,i)$, d-c gain of tandem connection of u^{th} through v^{th} element groups.

$$T(1,0) = 1.$$

$$g_1^*(t) = \frac{g_1(t)}{T(1,1)}.$$

$$h_1^*(t) = \frac{h_1(t)}{T(1,J)}.$$

$$k_i^*(t) = \frac{k_i(t)}{T(1, i-1)}, \quad i = 2, 3, \dots, J.$$

similar significance for $h_i^*(s)$ and $H_i^*(t)$, etc.

Ex - ensemble average of random variable x .

$\Phi(x_1, x_2) = \text{Ex}_1 x_2$, correlation function of random variables

x_1 and x_2 .

$u = \text{E}e(t)$, ensemble average of $e(t)$.

$\Phi_1(t_1, t_2)$ - Correlation function of $N_1(t_1)$ and $N_1(t_2)$.

$\phi_1(t_1, t_2)$ - Correlation function of $n_1(t_1)$ and $n_1(t_2)$.

$\Phi(t_1, t_2; \omega_1, \omega_2)$ - Correlation function of $e(t_1)$ and $e(t_2)$.

$\Phi(t_1, t_2) = \lim_{\substack{\omega_1 \rightarrow \infty \\ \omega_2 \rightarrow \infty}} \Phi(t_1, t_2; \omega_1, \omega_2)$, limiting form of correlation

function of $e(t_1)$ and $e(t_2)$.

$e(s; t)$ - value of $e(t)$ with all noise sources inactive.

$e(N_1; t)$ - value of $e(t)$ with all sources but $N_1(t)$ inactive.

$e(n_1; t)$ - value of $e(t)$ with all sources but $n_1(t)$ inactive.

* $\text{Ex}_1 x_2 - (\text{Ex}_1)(\text{Ex}_2)$ is called the covariance of x_1 and x_2 in statistical work.

The noise sources $N_1(t)$ and $n_1(t)$ have been discussed in Chapter IV. The $N_1(t)$ have been assumed to have the following properties: (1) They are Gaussian random processes applied to the system at $t = 0$, (2) $EN_1(t) = 0$, (3) $\phi_1(t_1, t_2)$ is given by

$$\phi_1(t_1, t_2) = A_1^2 |Ee_1(t_1)| \frac{\sin \omega_1(t_2 - t_1)}{t_2 - t_1} ; t_2 \geq t_1 ,$$

(4) ω_1 is very large so that the approximation that ω_1 approaches infinity can be used and (5) A given $N_1(t)$ is statistically independent of all other sources. Similar properties have been assumed for the $n_1(t)$, namely: (1) They are Gaussian random processes applied for a long time, (2) $En_1(t) = 0$, (3) $\psi_1(t_1, t_2)$ is given by

$$\psi(t_1, t_2) = B_1^2 \frac{\sin \omega_2(t_2 - t_1)}{t_2 - t_1} ; t_2 \geq t_1 ,$$

(4) ω_2 is very large so that as an approximation ω_2 approaches infinity, and (5) a given $n_1(t)$ is statistically independent of all other sources.

In developing the mathematical model for the biased estimating system in Chapter IV certain restrictions were placed on the system elements. It is expedient at this point in the discussion to explicitly state these restrictions and in fact impose several additional ones which lead to a tractable analysis.

It will be assumed that all element groups have transfer functions, $\bar{g}_1(s)$, which can be represented as the ratio of two polynomials in s , so that

$$\bar{g}_1(s) = \frac{p_1(s)}{q_1(s)} T(1,1) .$$

It will be further assumed that $\bar{g}_1(s)$ has no poles on the real frequency axis or in the right half s-plane, and that the degree of the polynomial $q_1(s)$ is greater than the degree of $p_1(s)$.

No poles or zeros are allowed at the origin of the s-plane for the following reason. Since the element groups are a part of a measurement system it seems reasonable to require that the step response of each element group reach a bounded non-zero steady-state value. That is to say, if a constant S which is to be measured is applied at $t = 0$ to any element group it is reasonable to assume that the element group output in the steady state would be proportional to S. This requirement may be stated mathematically as

$$\lim_{s \rightarrow 0} s \left[\frac{\bar{g}_1(s)}{s} \right] = \lim_{t \rightarrow \infty} \int_0^t g_1(x) dx = T(1,1) > 0$$

where use is made of the final value theorem for Laplace transforms.* The requirement

$$\lim_{s \rightarrow 0} \left[\bar{g}_1(s) \right] = T(1,1) > 0$$

is equivalent, for the class of functions being considered, to requiring that $\bar{g}_1(s)$ have no poles or zeros at the origin. These restrictions on the step response force the output of every system element to be bounded in the steady-state if its input is bounded. Thus

$$\int_0^{\infty} g_1(s)y(t-x) dx \leq M$$

* See for example Gardner and Barnes (16).

where $g_1(t)$ is the impulse response of any element group, $y(t)$ is a bounded input function, and M is a positive constant. For convenience in later work it will be assumed that the system elements of the diagram of Figure 9 have been chosen in such a way that no $\bar{g}_1(s)$ is a constant independent of s . This does not impose a restriction on the analysis.

It is clear that the application of the above restrictions to each element group limits the generality of the analysis. In fact, it would be less restrictive to apply the restrictions to only the overall measurement system transfer function. The restrictions have been applied as indicated, however, in order to obtain tractable results, which, as a matter of fact, will apply to a large majority of measurement systems.

The system represented by the diagram of Figure 9 will now be analyzed using the principle of superposition. Using the notations given above $e(t)$ may be expressed by the equation

$$e(t) = e(S;t) + \sum_{i=1}^J e(N_i;t) + \sum_{i=1}^J e(n_i;t) . \quad (17)$$

An expression for each of these component outputs will now be obtained.

Consider $e(S;t)$. In the notation which has been adopted $h_1(t)$ denotes the impulse response of the system from the point of application of S to the output. Thus use of the well known superposition integral^{*} and the fact that $S = 0$ for $t < 0$ yields for $e(S;t)$.

$$e(S;t) = S \int_0^t h_1(x) dx = S H_1(t) . \quad (18)$$

Now consider $e(N_i;t)$. In this case the superposition integral yields

^{*} See for example Gardner and Barnes (16).

$$e(N_1; t) = \int_0^t h_1(t - u) N_1(u) du . \quad (19)$$

Finally for the $e(n_1; t)$ the superposition integral yields

$$e(n_1; t) = \int_{-\infty}^t h_1(t - u) n_1(u) du . \quad (20)$$

Both $N_1(t)$ and $n_1(t)$ are random variables, thus the calculations to follow will have the objective of establishing the statistical properties of $e(t)$.

The first conclusion which may be drawn relative to the statistical properties of $e(t)$ is that it is a Gaussian random variable. This fact follows from two well known properties of the Gaussian random process, namely, (1) a Gaussian process remains Gaussian after passing through a combination of linear elements such as those considered here and (2) any finite linear combination, of Gaussian random variables, such as given by equation (17) above, is also a Gaussian random variable.*

A Gaussian random process has the special property that it is completely described by its mean and correlation function. These parameters will now be computed for $e(t)$ using ensemble averages since $e(t)$ is not a stationary process.

First consider $E e(t)$ beginning with equation (17). From the well known distributive property of the expected value, it follows that $E e(t)$ may be expressed as

$$E e(t) = E e(S; t) + \sum_{i=1}^J E e(n_i; t) + \sum_{i=1}^J E e(N_i; t) . \quad (21)$$

* See for example Laning and Battin (12), p. 156.

Equation (21) shows that $e(S;t)$ is a fixed function of time. Hence

$$E e(S;t) = S H_1(t) . \quad (22)$$

The expected value of $e(N_1;t)$ may be expressed as

$$E e(N_1;t) = E \int_0^t h_1(t-u) N_1(u) du . \quad (23)$$

The operation denoted by E represents an integration over all the members of an ensemble. For the class of functions being considered here for the integrand, the operation of taking the expected value can be interchanged with integration with respect to u .^{*} Thus equation (23) may be written

$$E e(N_1;t) = \int_0^t h_1(t-u) E N_1(u) du . \quad (24)$$

But since $E N_1(t) = 0$, it follows that $E e(N_1;t) = 0$. A similar argument shows that $E e(n_1;t) = 0$. Thus it follows that $E e(t)$ can be expressed as

$$E e(t) = S \int_0^t h_1(x) dx = S H_1(t) . \quad (25)$$

It is easily shown by using the same procedure as that above that $E e_1(t)$ is given by

$$E e_1(t) = S \int_0^t k_1(x) dx = S K_1(t) . \quad (26)$$

Now consider the correlation function of $e(t)$, $\phi(t_1, t_2; \omega_1, \omega_2)$

^{*}The conditions under which the order of integration can be interchanged are discussed by Hobson (17), among others.

given by

$$\Phi(t_1, t_2; \omega_1, \omega_2) = E e(t_1) e(t_2) \quad ; \quad t_1 \leq t_2 .$$

Using equation (17) this expression becomes

$$\begin{aligned} \Phi(t_1, t_2; \omega_1, \omega_2) = E & \left[e(S; t_1) + \sum_{i=1}^J e(N_i; t_1) + \sum_{i=1}^J e(n_i; t_1) \right] \\ & \left[e(S; t_2) + \sum_{i=1}^J e(N_i; t_2) + \sum_{i=1}^J e(n_i; t_2) \right] . \end{aligned} \quad (27)$$

By evaluating the product within the expected value operation it follows that the following terms must be considered:

a. $E e(S; t_1) e(S; t_2)$

b. $E e(S; t_1) \sum_{i=1}^J e(N_i; t_2) \quad ; \quad E e(S; t_1) \sum_{i=1}^J e(n_i; t_1)$

c. $E e(S; t_2) \sum_{i=1}^J e(N_i; t_1) \quad ; \quad E e(S; t_2) \sum_{i=1}^J e(n_i; t_1)$

d. $E \sum_{i=1}^J e(N_i; t_1) \sum_{i=1}^J e(n_i; t_2) \quad ; \quad E \sum_{i=1}^J e(N_i; t_2) \sum_{i=1}^J e(n_i; t_1)$

e. $E \sum_{i=1}^J e(n_i; t_1) \sum_{i=1}^J e(n_i; t_2) \quad ; \quad E \sum_{i=1}^J e(N_i; t_1) \sum_{i=1}^J e(N_i; t_2)$

Use of equation (22) gives

$$E e(S; t_1) e(S; t_2) = S^2 H_1(t_1) H_1(t_2)$$

for term a. A typical term of group b could be written

$$E e(S; t_1) e(N_1; t_2) = S \int_0^{t_1} h_1(x) dx \int_0^{t_2} h_1(t_2 - u) E N_1(u) du$$

since integration on u can be interchanged with taking the expected value.*

The fact that $E N_1(u)$ is zero makes this term zero. Further calculation shows that all terms of group b and c are zero. A typical term of group d could be written as

$$E e(N_s; t_1) e(n_1; t_2) = \int_0^{t_1} \int_{-\infty}^{t_2} h_s(t_1 - x_1) h_1(t_2 - x_2) E N_s(x_1) n_1(x_2) dx_1 dx_2 \quad (28)$$

since for the integrands being considered here the product of two integrals can be expressed as a double integral and this double integration can be interchanged with taking the expected value.**

The noise sources have been assumed to be statistically independent of the signal and each other. Thus in equation (28)

$$E N_s(x_1) n_1(x_2) = 0.$$

This fact and similar reasoning to that above leads to the conclusion that all the terms of group d are zero and that only terms of the type

$$E \sum_{i=1}^J e(n_i; t_1) e(n_i; t_2)$$

*The conditions under which this interchange is valid are discussed, for example, by Hobson (17).

**See for example Taylor (18) or Hobson (17).

in group e are non zero. Thus $\phi(t_1, t_2; \omega_1, \omega_2)$ is given by

$$\begin{aligned} \phi(t_1, t_2; \omega_1, \omega_2) &= S^2 H_1(t_1) H_1(t_2) \\ &+ \sum_{i=1}^J \int_0^{t_1} \int_0^{t_2} h_i(t_1 - x_1) h_i(t_2 - x_2) E N_1(x_1) N_1(x_2) dx_1 dx_2 \\ &+ \sum_{i=1}^J \int_{-\infty}^{t_1} \int_{-\infty}^{t_2} h_i(t_1 - x_1) h_i(t_2 - x_2) E n_1(x_1) n_1(x_2) dx_1 dx_2 . \end{aligned} \quad (29)$$

Introducing the correlation functions into equation (29) gives the following for $\phi(t_1, t_2; \omega_1, \omega_2)$

$$\begin{aligned} \phi(t_1, t_2; \omega_1, \omega_2) &= S^2 H_1(t_1) H_1(t_2) \\ &+ \sum_{i=1}^J \int_0^{t_1} \int_0^{t_2} h_i(t_1 - x_1) h_i(t_2 - x_2) S A_1^2 |K_1(x_1)| \frac{\sin \omega_1(x_2 - x_1)}{x_2 - x_1} dx_1 dx_2 \\ &+ \sum_{i=1}^J \int_{-\infty}^{t_1} \int_{-\infty}^{t_2} h_i(t_1 - x_1) h_i(t_2 - x_2) B_1^2 \frac{\sin \omega_2(x_2 - x_1)}{x_2 - x_1} dx_1 dx_2 . \end{aligned} \quad (30)$$

To aid in referring to the component parts of equation (30) introduce the notation

$$\phi(t_1, t_2; \omega_1, \omega_2) = I_1 + I_2(\omega_1) + I_3(\omega_2) . \quad (31)$$

The desired quantity is $\phi(t_1, t_2)$ given by

$$\phi(t_1, t_2) = \lim_{\substack{\omega_1 \rightarrow \infty \\ \omega_2 \rightarrow \infty}} \phi(t_1, t_2; \omega_1, \omega_2) .$$

Using equation (31), $\Phi(t_1, t_2)$ may be expressed as

$$\Phi(t_1, t_2) = I_1 + \lim_{\omega_1 \rightarrow \infty} I_2(\omega_1) + \lim_{\omega_2 \rightarrow \infty} I_3(\omega_2). \quad (32)$$

Consider $\lim_{\omega_1 \rightarrow \infty} I_2(\omega_1)$ as typical of the last two terms of equation (32).

This quantity is given by

$$\begin{aligned} \lim_{\omega_1 \rightarrow \infty} I_2(\omega_1) = \\ \lim_{\omega_1 \rightarrow \infty} S \sum_{i=1}^J A_i^2 \int_0^{t_1} \int_0^{t_2} h_i(t_1 - x_1) h_i(t_2 - x_2) |K_i(x_1)| \frac{\sin \omega_1(x_2 - x_1)}{x_2 - x_1} dx_1 dx_2. \end{aligned}$$

A change of variable

$$z = \omega_1(x_2 - x_1)$$

yields for $\lim_{\omega_1 \rightarrow \infty} I_2(\omega_1)$

$$\begin{aligned} \lim_{\omega_1 \rightarrow \infty} S \sum_{i=1}^J A_i^2 \int_0^{t_1} h_i(t_1 - x_1) |K_i(x_1)| \\ \int_{-\omega_1 x_1}^{\omega_1(t_2 - x_1)} h_i(t_2 - \frac{z}{\omega_1} - x_1) \frac{\sin z}{z} dz dx_1. \end{aligned}$$

The limiting process on ω_1 can be taken inside the first integral* to yield

* A formal statement of the conditions sufficient for this step to be valid is given for example by Hobson (17).

$$S \sum_{i=1}^J A_i^2 \int_0^{t_1} h_1(t_1 - x_1) |K_1(x_1)| \\ \left[\lim_{\omega_1 \rightarrow \infty} \int_{-\omega_1 x_1}^{\omega_1(t_2 - x_1)} h_1(t_2 - \frac{z}{\omega_1} - x_2) \frac{\sin z}{z} dz \right] dx_1 .$$

The integral within the brackets [] has the value

$$\pi h_1(t_2 - x_1) \quad ; \quad 0 < x_1 < t_2$$

$$\frac{\pi}{2} h_1(t_2 - x_1) \quad ; \quad x_1 = 0 \text{ or } x_1 = t_2$$

$$0 \quad ; \quad \text{otherwise} .$$

Thus $\lim_{\omega_1 \rightarrow \infty} I_2(\omega_1)$ can be expressed as

$$\lim_{\omega_1 \rightarrow \infty} I_2(\omega_1) = \pi S \sum_{i=1}^J A_i^2 \int_0^{t_1} h_1(t_1 - x_1) h_1(t_2 - x_1) |K_1(x_1)| dx_1$$

or in a more convenient form as

$$\pi S \sum_{i=1}^J A_i^2 \int_0^{t_1} h_1(u) h_1(u + t_2 - t_1) |K_1(t_1 - u)| du .$$

Following a similar calculation for $\lim_{\omega_2 \rightarrow \infty} I_3(\omega_2)$; $\Phi(t_1, t_2)$ can be expressed

as

$$\begin{aligned}
\phi(t_1, t_2) &= S^2 H_1(t_1) H_1(t_2) \\
&+ \pi S \sum_{i=1}^J A_i^2 \int_0^{t_1} h_i(u) h_i(u + t_2 - t_1) |K_i(t_1 - u)| du \quad (33) \\
&+ \pi \sum_{i=1}^J B_i^2 \int_0^{\infty} h_i(u) h_i(u + t_2 - t_1) du \quad ; \quad t_2 \geq t_1 .
\end{aligned}$$

Equations (25) and (33) are general expressions for the mean and correlation functions of the output of a group of suitably restricted measurement system elements such as represented by the diagram of Figure 9.

In order to use these general equations in a specific case it is necessary to specify the following:

1. The noise source constants A_i and B_i
2. The impulse responses $h_i(t)$
3. The impulse responses $k_i(t)$

Given these parameters, the mean and the correlation functions of $e(t)$ may be determined from the appropriate general equations in a straight forward way.

In later work the mean square value of $e(t)$ and the variance of $e(t)$ will be useful quantities. They can be obtained immediately from equation (33) as

$$E e(t)^2 = S^2 H_1(t)^2 + \pi S \sum_{i=1}^J A_1^2 \int_0^t h_1(u)^2 |K_1(t-u)| du$$

(34)

$$+ \sum_{i=1}^J B_1^2 \int_0^\infty h_1(u)^2 du$$

and

$$\text{Var } e(t) = \pi \sum_{i=1}^J \left[S A_1^2 \int_0^t h_1(u)^2 |K_1(t-u)| du + B_1^2 \int_0^\infty h_1(u)^2 du \right]. \quad (35)$$

The components of equation (34) for the variance of $e(t)$ can be given a physical interpretation. Consider the term

$$\pi \sum_{i=1}^J B_1^2 \int_0^\infty h_1(u)^2 du.$$

Parseval's equation* relates $\int_0^\infty h_1(u)^2 du$ to an integral over frequency as

$$\pi \int_0^\infty h_1(u)^2 du = \int_0^\infty |\bar{h}_1(\omega)|^2 d\omega.$$

Thus the last term of equation (34) may be written as

$$\sum_{i=1}^J B_1^2 \int_0^\infty |\bar{h}_1(\omega)|^2 d\omega.$$

The constants B_1^2 may be interpreted as the energy spectral density of

*A typical discussion of this is found in Laning and Battin (12).

the $n_i(t)$ noise sources. Thus

$$\sum_{i=1}^J B_i^2 \int_0^{\infty} |\bar{h}_i(\omega)|^2 d\omega$$

represents the total energy transferred from the noise sources $n_i(t)$ to the system output in the time required for the system to reach the steady-state with only the $n_i(t)$ sources applied. It is clear that the value of

$$\sum_{i=1}^J B_i^2 \int_0^{\infty} |\bar{h}_i(\omega)|^2 d\omega$$

is independent of time, depending on the constants B_i^2 and the transfer functions of the system.

The term

$$\pi S \sum_{i=1}^J A_i^2 \int_0^t h_i(u)^2 |K_i(t-u)| du$$

has the dimensions of energy, and can be interpreted as the energy transferred from the noise sources $N_i(t)$ to the system output in the time interval $(0, t)$.

CHAPTER VI

FORMULATION AND ANALYSIS OF A MATHEMATICAL MODEL FOR THE OUTPUT METER

In the general measurement system model discussed in Chapter II the final element is a device, termed the output meter, which produces a reading or display appropriate to a particular quantity being measured. For the restricted class of measurement systems being considered here the output meter should indicate a constant value in the steady-state for ideal noise-free operation. Illustrative of the devices which could perform this operation are the D'Arsonval meter, a closed-loop position servo such as used with electronic recorders, and a digital display. Any device of this type will be subject to internal noise, and will introduce inertia in the same fashion as the elements of the transducer and amplification system. In the mathematical model being formulated the elements necessary to account for this noise and inertia of the output meter will be included in the amplification system. Thus the component termed "output meter" in the mathematical model will have only those properties associated with an inertia-less indicating device. It is the purpose of this chapter to formulate and analyze a mathematical model for an output meter having these properties. The mathematical model will be formulated specifically for devices using a scale and pointer, although it could be extended to other types of indicators.

Two different types of errors will be attributed to the output meter. The first error which will be called "movement error" arises because the

actual instantaneous indicator position can differ from that which would be proportional to the output meter input. Static friction in the indicator mechanism of open loop indicators and parallax in all type indicators are representative of the mechanisms producing this type of error.

The second type of error could be termed "round-off" error. This error arises because of the fact that no physical scale and pointer can be read to an arbitrarily large number of significant figures.

The mathematical models chosen to represent movement error and round-off error will now be discussed. These models have been chosen so that they are tractable and conform in a general way to what would be expected for each type of physical mechanism. Movement error will be assumed to be a Gaussian random variable with zero mean. Its variance will be determined by the characteristics of a particular output meter. This representation is generally consistent with the characteristics to be expected of movement error. In particular, errors of this type would be expected to average to zero in a large number of readings and large errors would be expected to occur less frequently than small errors. Movement error will be assumed to be independent of the signal and all other system errors.*

Movement error depends on the mechanical properties of a particular indicating device. In many practical output meters it is common practice to employ a single indicating device with other appropriate components so as to achieve a number of different sensitivities. In such cases the

* The term independence is used here in its mathematical sense to mean that if α and θ are two distinct random variables, having probability distribution functions $F_1(x)$ and $F_2(y)$ and a joint distribution $F(x,y)$ they are independent if $F(x,y) = F_1(x) F_2(y)$.

mechanical movement error is the same for several meter sensitivities and it is convenient to express this error as a fraction, E_m , of full scale. Thus for an indicator having several sensitivities the movement error, denoted by ϵ_m in input units, is given by

$$\epsilon_m = E_m V \quad (35)$$

where V is the calibration of mechanical full scale in input units. Both ϵ_m and E_m are random variables. For a given indicating device the statistical properties of E_m are fixed while ϵ_m can be varied by changing the sensitivity of the indicating device. Of course, E_m will vary for different indicating devices.

Considerations to be presented in Chapter VII lead to a condition such that the effective indicator sensitivity changes with time. In such a case ϵ_m is a function of time, and the notation

$$\epsilon_m(t) = V(t) E_m \quad (36)$$

will be used. In this same chapter it is necessary to consider the quantity

$$E \epsilon_m(t_i) \epsilon_m(t_j) .$$

This quantity will be assumed to be negligibly small if t_i is not equal to t_j . This assumption is equivalent to assuming that there is no correlation between the values of movement error taken at different times.

It will be assumed that movement error is an additive type error. Thus the actual position of an output meter indicator will differ from an

amount equal to output meter input by an amount equal to the magnitude of ϵ_m . This fact can be expressed by the equation

$$r_2(t) = r_1(t) + \epsilon_m \quad (37)$$

where all three variables are in input units, $r_1(t)$ being the output meter input, ϵ_m the movement error referred to the input, and $r_2(t)$ the actual indicator position referred to the input.

The round-off error arises because there is a smallest scale division which can be reliably identified. For example, consider reading a particular scale and pointer. For a fixed position of the pointer there would exist a smallest interval on the scale in which the pointer could be determined to indicate with probability one. The pointer could not be said to indicate in a smaller interval with certainty, i.e., the probability that the pointer indicates in a smaller interval would be less than one. This smallest scale division which can be reliably identified will be called the critical interval.

The remarks above can be illustrated by the scale and pointer of Figure 10. A study of this figure reveals that pointer 1 certainly indicates in the range 6-7 with probability 1. Furthermore there is little doubt that the pointer indicates in the range 6.5 to 7. However, as the size of the interval is diminished it becomes more and more difficult to state with certainty that the pointer indicates in a given interval. Although the exact size of the critical interval may depend to some extent on the observer, it is clear that in this example it is close to 0.25, that is, the pointer indicates in the range 6.5 to 6.75 with probability 1 and this statement could not be made for a much smaller interval. The

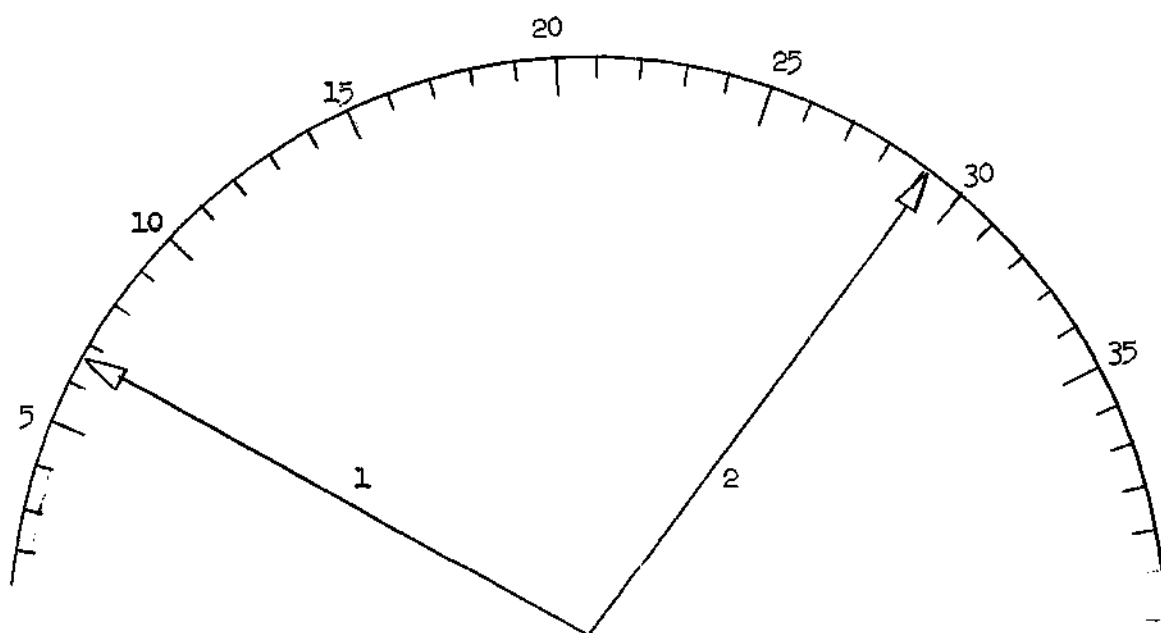


Figure 10 Illustration of a Scale and Pointer

same conclusion concerning the size of the critical scale division would be reached by examining pointer 2.

Consideration of the problem of reading a scale and pointer makes two facts apparent. First, the critical scale division depends on the physical properties of the scale and pointer, and second the critical scale division is independent of the actual position of the pointer.

As was the case for movement error, it is expedient to express the round-off error in input units by the equation

$$\epsilon_r = E_r V \quad (38)$$

where E_r is mechanical round-off error expressed as a fraction of full scale, ϵ_r is round-off error in input units, and V is the calibration of full scale in input units. This equation places in evidence the fact that for a single indicator having several sensitivities, ϵ_r depends on V . Both ϵ_r and E_r are random variables having statistical properties dependent on the physical properties of the scale and pointer.

In Chapter VII a time dependent sensitivity will be introduced so that ϵ_r will depend on time as given by

$$\epsilon_r(t) = V(t) E_r \quad (39)$$

As a matter of fact a later equation will show that ϵ_r and E_r are inherently time dependent. However, since this time dependence is secondary in this chapter the notation of simply ϵ_r for the round-off error will be continued. As a further comment on the properties of ϵ_r , it will be assumed in all cases that its ensemble statistics are independent of time. Thus for example

$$E E_r(t_i) = E E_r(t_j) \quad , \quad \text{all } i, j.$$

In Chapter VII the assumption will be made that

$$E \epsilon_r(t_i) \epsilon_r(t_j)$$

is negligibly small.

A mathematical model accounting for this type of round-off error, occurring for a linear scale, will now be given. The meter reading, which will be denoted $R(t)$, can be expressed as*

$$R(t) = \begin{cases} n\gamma + \gamma/2 + \eta & , \quad n\gamma \leq r_2(t) < (n+1)\gamma \\ 0 & , \quad r_2(t) < 0 \\ N\gamma & , \quad N\gamma \leq r_2(t) \end{cases} \quad (40)$$

where γ^{**} is the critical scale division in input units; $n = 0, 1, 2, \dots, N-1$;

* While the choice of this mathematical expression for $R(t)$ is considered to best represent the physical situation, it can be shown that the results obtained using this model differ only in unimportant details from either of the choices

$$R(t) = n\gamma \quad , \quad n\gamma \leq r_2(t) < (n+1)\gamma$$

$$R(t) = n\gamma + \gamma/2 \quad , \quad n\gamma \leq r_2(t) < (n+1)\gamma$$

The former is used by Grenander (19); the latter by Cramer (8) in discussing similar problems.

** It should be noted that γ , being in input units, is a function of the meter sensitivity for a fixed meter movement. Thus in cases where the meter sensitivity is a function of time γ will be also. The symbol Γ will be used for the critical scale division in output units.

$N\gamma$ is the maximum possible meter reading equal to V ; and η is a random variable assuming each of the values $\gamma/2$ and $-\gamma/2$ with probability $1/2$. η is assumed to be statistically independent of $r_1(t)$ and $r_2(t)$. For convenience the output meter reading, $R(t)$, is expressed here in input units. It is clear, however, that $R(t)$ could be expressed easily in output units if this were desirable.

To illustrate the nature of $R(t)$ consider a sequence of readings taken with a scale and pointer represented by the mathematical model at equally spaced time intervals on the particular $r_2(t)$ function shown in the solid curve of Figure 11. For purposes of this illustration N is 10 and γ is the interval shown in the figure. A possible sample function from the $R(t)$ ensemble is plotted as the broken curve in Figure 11. In drawing this curve $R(t)$ is artificially assumed to have the constant value $R(t_1)$ in the interval $t_1 \leq t < t_1 + 1$.

In this mathematical model for the output meter the range of the meter is from 0 to $N\gamma$. Thus the output is restricted to non-negative numbers. This is not a severe restriction on the model, however, since a sign reversal could be made to measure a negative quantity.

It is common practice to restrict the operation of a measurement system element to its linear range. Thus "linear range" for the output meter will be defined mathematically and further discussion will be restricted to operation in this range.*

In the notation adopted above, $r_1(t)$ denotes the input to the output meter. This quantity will in general be a random variable. The

* In later chapters when the complete measurement system is considered, it will be assumed that the complete system is limited in linear range by the properties of the output meter. Thus the definition of linear range to be given here for the output meter will be used for the complete measurement system.

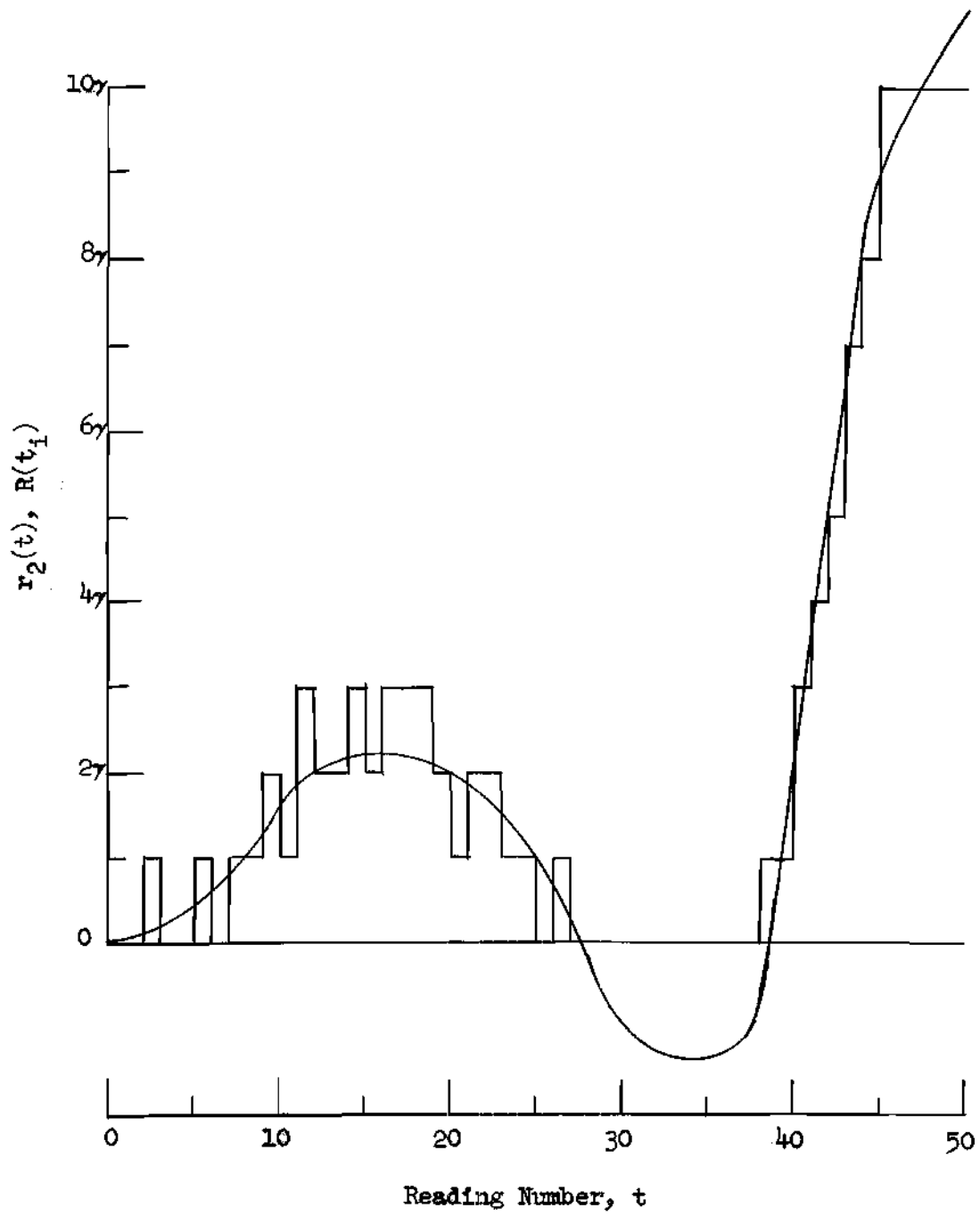


Figure 11. A Representative Sequence of Meter Readings

symbols σ_{r_1} and μ will be used to denote its standard deviation and mean. $r_2(t)$ has been expressed by equation (37) as

$$r_2(t) = r_1(t) + \epsilon_m.$$

Since ϵ_m has a mean of zero, the mean of $r_2(t)$ is equal to μ . The standard deviation of $r_2(t)$ will be denoted σ_{r_2} .

It seems reasonable to require that for the output meter to be in its linear range $r_2(t)$ should be between 0 and $N\gamma$. However, since $r_2(t)$ is a random variable with a potential range of the whole real line it is not possible to state precise limits on the value of $r_2(t)$. Since it is possible to make probability type statements relative to the value of $r_2(t)$, "linear range" will be defined so that $r_2(t)$ is in the range 0 to $N\gamma$ with high probability. Such an expedient effectively disregards the small number of experiments in which $r_2(t)$ would differ, for example, from its mean by a large multiple of its standard deviation. Specifically, "linear range" will be defined in terms of μ as follows:

"An output meter will be operating in its linear range if the inequality

$$4\sigma_{r_2} < \mu < N\gamma - 4\sigma_{r_2}$$

is satisfied." This definition places $r_2(t)$ in the range 0 to $N\gamma$, with probability at least 0.999936 if $r_2(t)$ has a Gaussian distribution function and if the output meter is in its linear range. Figure 12 is a plot of the two "limiting" distributions for a Gaussian $r_2(t)$ with operation restricted to the "linear range."

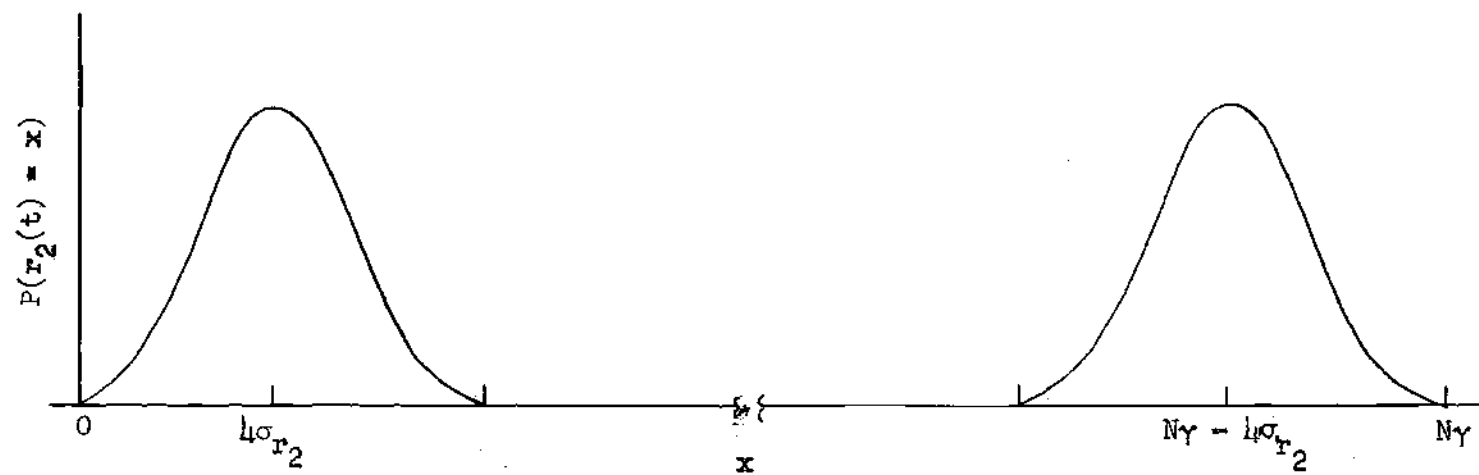


Figure 12 Extreme Probability Density Function for $r_2(t)$ for
Operation in the Linear Range

Return now to a consideration of the statistical properties of the output meter reading $R(t)$. $R(t)$ was defined by equation (40). This definition of $R(t)$ makes it a variable taking on $N + 1$ discrete values. These $N + 1$ numbers constitute the sample space of the measurement system.

The probability, p_1 , that $R(t) = i\gamma$ for $i = 1, 2, \dots, N - 1$ can be computed as follows. The event $(R(t) = i\gamma)$ can be expressed as

$$(R(t) = i\gamma) = (i\gamma \leq r_2(t) < (i + 1)\gamma) \quad (\eta = \gamma/2)$$

$$U((i - 1)\gamma \leq r_2(t) < i\gamma) \quad (\eta = \gamma/2)$$

where $(A) (B)$ denotes the intersection of the events (A) and (B) and $(A) \cup (B)$ denotes the union of the events (A) and (B) . η is assumed to be statistically independent of the random variable $r_2(t)$. Hence, the probability of each intersection above is the product of the corresponding probabilities. Since the probability of the intersection of $(i\gamma \leq r_2(t) < (i + 1)\gamma) ((i - 1)\gamma \leq r_2(t) < i\gamma)$ is zero, p_1 can be expressed as

$$p_1 = P \left[(R(t) = i\gamma) \right] = \frac{1}{2} P \left[(i\gamma \leq r_2(t) < (i + 1)\gamma) \right] \\ + \frac{1}{2} P \left[((i - 1)\gamma \leq r_2(t) < i\gamma) \right].$$

The probability of the event $(i\gamma \leq r_2(t) < (i + 1)\gamma)$ can be expressed as

$$P(i\gamma \leq r_2(t) < (i + 1)\gamma) = \int_{i\gamma}^{(i+1)\gamma} dF_2(x)$$

for $i = 1, 2, \dots, N - 1$ where $F_2(x)$ denotes the probability distribution

function of $r_2(t)^*$. A similar integral between the limits of $(i-1)\gamma$ and $i\gamma$ gives the probability of the event $((i-1)\gamma \leq r_2(t) < i\gamma)$ for $i = 1, 2, \dots, N-1$. Thus the probability that $R(t) = i\gamma$ is given by

$$p_i = P(R(t) = i\gamma) = \frac{1}{2} \int_{(i-1)\gamma}^{(i+1)\gamma} d F_2(x)$$

for $i = 1, 2, \dots, N-1$. A similar calculation yields expressions for p_0 and p_N . These expressions are

$$p_0 = \int_{-\infty}^0 d F_2(x) + \frac{1}{2} \int_0^{\gamma} d F_2(x) \quad (41)$$

and

$$p_N = \int_{N\gamma}^{\infty} d F_2(x) + \frac{1}{2} \int_{(N-1)\gamma}^{N\gamma} d F_2(x) . \quad (42)$$

Note that for an instrument operating in its linear range the contribution of the integrals

$$\int_{-\infty}^0 d F_2(x) \quad \text{and} \quad \int_{N\gamma}^{\infty} d F_2(x) \quad \text{to } p_0 \text{ and } p_N$$

are extremely small. For example, if $F_2(x)$ is a Gaussian probability distribution function the definition of linear range makes each of these integrals less than 3.2×10^{-5} .

* In most of the later work $F_2(x)$ will be taken to be the Gaussian probability distribution function.

The mean of $R(t)$ can be calculated from expression

$$E R(t) = \sum_{i=0}^N i \gamma p_i \quad (43)$$

while the mean square deviation from μ is given by

$$E [R(t) - \mu]^2 = \sum_{i=0}^N (i\gamma)^2 p_i - \mu^2. \quad (44)$$

If $F_2(x)$ is a Gaussian distribution function, the integrals expressing the p_i cannot be evaluated in closed form. They can, however, be expressed in terms of $\lambda(x)$ given by

$$\lambda(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-y^2/2} dy$$

which is tabulated.* The expressions in terms of $\lambda(x)$ are the following:

$$p_i = \frac{1}{2} \left[\lambda \left[\frac{(i+1)\gamma - \mu}{\sigma_{r_2}} \right] - \lambda \left[\frac{(i-1)\gamma - \mu}{\sigma_{r_2}} \right] \right] \quad i = 1, 2, \dots, N-1$$

$$p_0 = \lambda \left[\frac{-u}{\sigma_{r_2}} \right] + \frac{1}{2} \left[\lambda \left[\frac{\gamma - u}{\sigma_{r_2}} \right] - \lambda \left[\frac{-u}{\sigma_{r_2}} \right] \right] \quad (45)$$

$$p_N = 1 - \lambda \left[\frac{N\gamma - \mu}{\sigma_{r_2}} \right] + \frac{1}{2} \left[\lambda \left[\frac{N\gamma - \mu}{\sigma_{r_2}} \right] - \lambda \left[\frac{(N-1)\gamma - \mu}{\sigma_{r_2}} \right] \right].$$

As indicated above, it is often expedient to introduce a round-off

* See for example Fisher (20).

error term, ϵ_r , so that $R(t)$ is given by

$$R(t) = r_2(t) + \epsilon_r \quad (46)$$

From this equation it follows that $E R(t)$ and $E (R(t) - \mu)^2$ are given by

$$E R(t) = E r_2(t) + E \epsilon_r = \mu + E \epsilon_r \quad (47)$$

and

$$E (R(t) - \mu)^2 = \sigma_{r_2}^2 + 2E r_2(t) \epsilon_r + E \epsilon_r^2 - 2\mu E \epsilon_r \quad (48)$$

Consider the random variable ϵ_r . This quantity can be expressed by the equation

$$\epsilon_r = \begin{cases} n\gamma + \frac{\gamma}{2} + \eta - r_2(t) & ; n\gamma \leq r_2(t) < (n+1)\gamma \\ -r_2(t) & ; r_2(t) < 0 \\ N\gamma - r_2(t) & ; N\gamma \leq r_2(t) \end{cases} \quad (49)$$

where $n = 0, 1, 2, \dots, N-1$.

By a straightforward, although tedious, calculation, the probability distribution function, $H(x)$, of ϵ_r may be expressed approximately in terms of $F_2(x)$ as

$$H(x) = \begin{cases} 1 & ; x \geq \gamma \\ \frac{1}{2} + \frac{1}{2} \sum_{n=0}^{N-1} \int_{(n+1)\gamma-x}^{(n+1)\gamma} dF_2(y) & ; 0 \leq x < \gamma \\ \frac{1}{2} \sum_{n=0}^{N-1} \int_{n\gamma-x}^{(n+1)\gamma} dF_2(y) & ; -\gamma \leq x < 0 \\ 0 & ; x < -\gamma \end{cases} \quad (50)$$

In arriving at this result it has been assumed that the integrals

$$\int_{-\infty}^0 dF_2(x) \quad \text{and} \quad \int_{Ny}^{\infty} dF_2(x)$$

have negligible values, a fact that is justified since the output meter will be used only in its linear range. In examining $H(x)$ it should be noted that subject to the approximation cited above

$$1 - H(0) = \int_0^{\gamma} dH(x) = \frac{1}{2}$$

and

$$H(0) = \int_{-\gamma}^0 dH(x) = \frac{1}{2} .$$

In words, ϵ_r is between $-\gamma$ and γ with certainty and has equal probability of lying in the two intervals $[-\gamma, 0]$ and $[0, \gamma]$.

In terms of $H(x)$, $E \epsilon_r$ may be expressed as

$$E \epsilon_r = \int_{-\infty}^{\infty} x dH(x) = \int_{-\gamma}^{\gamma} x dH(x) . \quad (51)$$

It is convenient to express

$$\int_{-\gamma}^{\gamma} x dH(x)$$

as

$$\int_{-\gamma}^{\gamma} x dH(x) = \int_0^{\gamma} x dH(x) + \int_{-\gamma}^0 x dH(x)$$

and to note that each of the integrals on the right hand side of the above equation are bounded as follows

$$0 \leq \int_0^{\gamma} x dH(x) \leq \int_0^{\infty} x dH(x) = \frac{\gamma}{2}$$

$$-\frac{\gamma}{2} = \int_{-\infty}^0 x dH(x) \leq \int_{-\gamma}^0 x dH(x) \leq 0.$$

Making use of these results it follows that $E \epsilon_r$ is bounded above and below by $\gamma/2$ and $-\gamma/2$ respectively, thus

$$-\frac{\gamma}{2} \leq E \epsilon_r \leq \frac{\gamma}{2} \quad . \quad (52)$$

An example will serve to illustrate that these bounds are in fact the least upper bound and the greatest lower bound for ϵ_r for the cases being considered. Consider the case that $r_2(t)$ has a fixed value, \underline{a} , which lies in the interval $[0, (N-1)\gamma]$. Such a choice for $r_2(t)$ would result in a probability distribution function $F_2(x)$ given by

$$F_2(x) = \begin{cases} 0 & ; \quad x < \underline{a} \\ 1 & ; \quad x \geq \underline{a} \end{cases}$$

The constant \underline{a} must lie in some interval of length γ in the larger interval $[0, (N-1)\gamma]$. Denote this interval by $[n\gamma, (n+1)\gamma]$ and let \underline{a}' be defined as

$$a' = a - [n] \gamma$$

where $[n]$ denotes the largest integer in a' . It follows that a' lies in the interval $[0, \gamma]$. Now the particular $F_2(x)$ chosen for this example may be used in the expression of equation (50) for $H(x)$ to yield in this case

$$H(x) = \begin{cases} 1 & ; \quad x \geq \gamma - a' \\ \frac{1}{2} & ; \quad -a' \leq x < \gamma - a' \\ 0 & ; \quad x < -a' \end{cases}$$

Thus $E \in_r$ is given by

$$E \in_r = \int_{-\gamma}^{\gamma} x dH(x) = \frac{\gamma}{2} - a'.$$

Three particular choices for \underline{a} serve to illustrate the extremes of $E \in_r$ in this example. Consider the values for \underline{a} given below with the corresponding values for a' :

$$a = [n] \gamma \quad ; \quad a' = 0$$

$$a = [n] \gamma + \frac{\gamma}{2} \quad ; \quad a' = \frac{\gamma}{2}$$

$$a = ([n] + 1)\gamma \quad ; \quad a' = \gamma.$$

For each of these cases $E \in_r$ may be determined as:

$$a = [n] \gamma \quad ; \quad E \epsilon_r = \frac{\gamma}{2}$$

$$a = [n] \gamma + \frac{\gamma}{2} \quad ; \quad E \epsilon_r = 0$$

$$a = ([n] + 1) \gamma \quad ; \quad E \epsilon_r = -\frac{\gamma}{2} .$$

Note that since a' is arbitrary $[n]$ can be any interger $n = 1, 2, \dots, N - 1$.

Thus in this example both the least upper bound and the greatest lower bound are realized for different choices of \underline{a} .

Now consider $E \epsilon_r^2$. This quantity may be expressed as

$$E \epsilon_r^2 = \int_{-\gamma}^{\gamma} x^2 dH(x) .$$

Bounds on $E \epsilon_r^2$ can be established as follows: In order to place in evidence the least upper bound and the greatest lower bound it is expedient to introduce the following notation

$$\begin{aligned} H_1(x) &= \begin{cases} H(x) & ; 0 \leq x < \gamma \\ 0 & ; \text{otherwise} \end{cases} \\ H_2(x) &= \begin{cases} H(x) & ; -\gamma \leq x < 0 \\ 0 & ; \text{otherwise} \end{cases} \end{aligned} \quad (53)$$

Thus $E \epsilon_r^2$ can be expressed as

$$E \epsilon_r^2 = \int_{-\gamma}^{\gamma} x^2 dH(x) = \int_0^{\gamma} x^2 dH_1(x) + \int_{-\gamma}^0 x^2 dH_2(x) .$$

In the integral involving $H_1(x)$ make the change of variable $x = y + \frac{\gamma}{2}$.

In the integral involving $H_2(x)$ make the change of variable $x = y - \frac{\gamma}{2}$.

This yields for $E \epsilon_r^2$ the expression

$$E \epsilon_r^2 = \int_{-\gamma/2}^{\gamma/2} (y + \frac{\gamma}{2})^2 dH_1(y + \frac{\gamma}{2}) + \int_{-\gamma/2}^{\gamma/2} (y - \frac{\gamma}{2})^2 dH_2(y - \frac{\gamma}{2}) .$$

From equations (50) and (53) it can be noted that integration with respect to $H_1(y + \frac{\gamma}{2})$ is equivalent to integration with respect to $H_2(y - \frac{\gamma}{2})$. Thus $E \epsilon_r^2$ can be expressed for example as

$$E \epsilon_r^2 = 2 \int_{-\gamma/2}^{\gamma/2} (y^2 + \frac{\gamma^2}{4}) dH_1(y + \frac{\gamma}{2}) .$$

But since

$$\frac{\gamma^2}{4} \leq y^2 + \frac{\gamma^2}{4} \leq \frac{\gamma^2}{2}$$

on the interval $-\frac{\gamma}{2} \leq y \leq \frac{\gamma}{2}$, it follows that

$$\frac{\gamma^2}{2} \int_{-\gamma/2}^{\gamma/2} dH_1(y + \frac{\gamma}{2}) \leq 2 \int_{-\gamma/2}^{\gamma/2} (y^2 + \frac{\gamma^2}{4}) dH_1(y + \frac{\gamma}{2}) \leq \gamma^2 \int_{-\gamma/2}^{\gamma/2} dH_1(y + \frac{\gamma}{2}) .$$

A further change of the dummy variable of integration so that $z = y + \frac{\gamma}{2}$ yields

$$\frac{\gamma^2}{2} \int_0^{\gamma} dH_1(z) \leq E \epsilon_r^2 \leq \gamma^2 \int_0^{\gamma} dH_1(z) .$$

But

$$\int_0^{\gamma} dH_1(z) = \int_0^{\gamma} dH(z) = \frac{1}{2} .$$

Therefore $E \epsilon_r^2$ satisfies the inequality

$$\frac{\gamma^2}{4} \leq E \epsilon_r^2 \leq \frac{\gamma^2}{2} . \quad (54)$$

Consider again the example used above in examining $E \epsilon_r$. In this case $E \epsilon_r^2$ can be expressed as

$$E \epsilon_r^2 = \frac{\gamma^2}{2} - a^2\gamma + (a^2)^2$$

The values of a considered above yield for $E \epsilon_r^2$ the following values:

$$a = n\gamma \quad ; \quad E \epsilon_r^2 = \frac{\gamma^2}{2}$$

$$a = n\gamma + \frac{\gamma}{2} \quad ; \quad E \epsilon_r^2 = \frac{\gamma^2}{4}$$

$$a = (n + 1)\gamma \quad ; \quad E \epsilon_r^2 = \frac{\gamma^2}{2}$$

Thus this example illustrates that $\gamma^2/2$ and $\gamma^2/4$ are in fact the least upper bound and greatest lower bound on $E \epsilon_r^2$.

To summarize this example, a fixed input, a , satisfying the inequalities

$$0 \leq n\gamma \leq a \leq (n + 1)\gamma \leq N\gamma$$

is applied to the output meter. The output meter reading, $R(t)$, is thus either $n\gamma$ or $(n + 1)\gamma$. $E R(t)$ is given by

$$E R(t) = n\gamma + \frac{\gamma}{2} .$$

The error will depend on the exact value of \underline{a} . $E \epsilon_r$ and $E (R(t) - a)^2$ (i.e., the bias error and mean square error) are as follows for particular choices of \underline{a}

$$a = n\gamma ; E \epsilon_r = \frac{\gamma}{2} ; E (R(t) - a)^2 = \frac{\gamma^2}{2}$$

$$a = n\gamma + \frac{\gamma}{2} ; E \epsilon_r = 0 ; E (R(t) - a)^2 = \frac{\gamma^2}{4}$$

$$a = (n + 1)\gamma ; E \epsilon_r = -\frac{\gamma}{2} ; E (R(t) - a)^2 = \frac{\gamma^2}{2} .$$

Most frequently the output meter will be used to estimate its input which is unknown a priori. Thus in this example if a constant unknown input, \underline{a} , is applied to the output meter, $E R(t)$ which could be estimated from physical experiments, would be an estimate of \underline{a} with a bias error bounded in magnitude by $\gamma/2$. The mean square deviation of $R(t)$ from \underline{a} could never be less than $\gamma^2/4$ and would not exceed $\gamma^2/2$. Tchebycheff's inequality, the results of which are plotted in Figure 3, Chapter III, could be used to state with what probability a single reading, $R(t_1)$, would be in a specified range about \underline{a} .

In work with the output meter it is expected that the mean square deviation of the output meter reading from μ will be the quantity of greatest utility. This quantity is expressed by equation (48) as

$$E (R(t) - \mu)^2 = \sigma_{r_2}^2 + E \epsilon_r^2 + 2 E r_2(t) \epsilon_r - \mu E \epsilon_r .$$

The final term in this expression depends on the joint distribution function of $r_2(t)$ and ϵ_r . Thus in order to proceed with the analysis it is

necessary to consider this joint distribution function which will be denoted $G(x, y)$.

By definition $G(x, y)$ is the probability that ϵ_r is less than or equal to x while at the same time $r_2(t)$ is less than or equal to y . With this definition as a starting point equation (50) for $H(x)$ can be modified in a straightforward way to obtain for $G(x, y)$ the expression

$$G(x, y) = \begin{cases} F_2(y) & ; \quad y \leq x \\ \frac{1}{2} + \frac{1}{2} \sum_{n=0}^{\left[\frac{y}{\gamma}\right]-1} \frac{(n+1)\gamma}{(n+1)\gamma-x} \int_{(n+1)\gamma-x}^y dF_2(y) + \int_{\left[\frac{y}{\gamma}\right]\gamma}^y dF_2(y) & ; \quad 0 \leq x < y \\ \frac{1}{2} \sum_{n=0}^{\left[\frac{y}{\gamma}\right]-1} \int_{n\gamma-x}^{(n+1)\gamma} dF_2(y) + \int_{\left[\frac{y}{\gamma}\right]\gamma}^y dF_2(y) & ; \quad -y \leq x < 0 \\ 0 & ; \quad x < -y. \end{cases} \quad (55)$$

This expression is based on the assumption that the output meter is in its linear range.

A consideration of a physical output meter leads to the conclusion that in most cases of importance the round-off error does not depend on the actual input to the output meter provided the device is in its linear range and provided that γ is not appreciably larger than σ_r . Thus the statistical independence of $r_2(t)$ and ϵ_r will be taken as a physical assumption. It will now be demonstrated that this assumption is not inconsistent with the mathematical model established above for the output meter.

From a mathematical point of view the random variables x and y are independent if

$$G(x, y) = H(x) F_2(y) .$$

Equation (55) shows that if $\gamma \leq x$

$$G(x, y) = F_2(y) .$$

Therefore since $H(x) = 1$ for $\gamma \leq x, \epsilon_r$ and $r_2(t)$ are independent in this region. For $x < -\gamma$

$$G(x, y) = 0$$

and $H(x) = 0$. Therefore the variables are independent in this region.

Finally if $x = 0$ equation (55) gives

$$G(x, y) = \frac{1}{2} F_2(y) .$$

Thus since $H(0) = \frac{1}{2}$, $r_2(t)$ and ϵ_r are independent for this value of x .

Thus the mathematical model gives rigorously

$$G(x, y) = H(x) F_2(y)$$

for $\gamma \leq x$, $x < -\gamma$ and $x = 0$ for all y . Hence the assumption that $r_2(t)$ and ϵ_r are statistically independent for small γ is not inconsistent with the mathematical model, although it should be emphasized that the statistical independence of $r_2(t)$ and ϵ_r has not been proven as a consequence of the mathematical model.

If $r_2(t)$ and ϵ_r are assumed to be independent the expression for

$E (R(t) - \mu)^2$ reduces to

$$E(R(t) - \mu)^2 = \sigma_{r_2}^2 + E \epsilon_r^2 . \quad (56)$$

It should be noted that all of the statistics of $R(t)$ and ϵ_r depend on the distribution function, $F_2(x)$, of $r_2(t)$. In most cases of interest the mean, μ , of $r_2(t)$ and hence one parameter of $F_2(x)$ is not known a priori. In fact in most cases this is the parameter which will be estimated by the measurement. Thus in such cases it is impossible, a priori, to calculate exact values for $E R(t)$, $E \epsilon_r$, etc. The bounds determined above, however, apply for any value of μ and thus will be useful in a priori calculations.

The above development of a mathematical model for the output meter will now be summarized. The output meter reading in input units, $R(t)$, is given by

$$R(t) = r_1(t) + \epsilon_m + \epsilon_r \quad (57)$$

where ϵ_m is movement error, ϵ_r is round-off error, and $r_1(t)$ is the input to the output meter. ϵ_m and ϵ_r are random variables. ϵ_m has been assumed to have a Gaussian distribution with mean zero and standard deviation, σ_m , determined by the mechanical properties of the meter movement. ϵ_r has a probability distribution function, $H(x)$, given by

$$H(x) = \begin{cases} 1 & ; \gamma \leq x \\ \frac{1}{2} + \frac{1}{2} \sum_{n=0}^{N-1} \int_{(n+1)\gamma-x}^{(n+1)\gamma} dF_2(y) & ; 0 \leq x < \gamma \\ \frac{1}{2} \sum_{n=0}^{N-1} \int_{n\gamma-x}^{(n+1)\gamma} \frac{(n+1)\gamma}{n\gamma-x} & ; -\gamma \leq x < 0 \\ 0 & ; x < -\gamma \end{cases} \quad (58)$$

where γ is the critical scale division in input units and $F_2(x)$ is the probability distribution function of $r_2(t)$ which is related to $r_1(t)$ by the equation

$$r_2(t) = r_1(t) + \epsilon_m. \quad (59)$$

The expression for $H(x)$ and all later results are based on the fact that the output meter is operated in its linear range, which requires that the expected value of the input lie in the range $4\sigma_{r_2} \leq \mu < N\gamma - 4\sigma_{r_2}$, where σ_{r_2} is the standard deviation of $r_2(t)$.

The expected value of $R(t)$ is given by

$$E R(t) = E r_1(t) + E \epsilon_r. \quad (60)$$

$E r_1(t)$ has been denoted by μ and $E \epsilon_r$ can be expressed as

$$E \epsilon_r = \sum_{n=0}^{N-1} \int_{-\gamma/2}^{\gamma/2} y dH(y).$$

Thus $E R(t)$ can be expressed as

$$E R(t) = \mu + \sum_{n=0}^{N-1} \int_{-\gamma/2}^{\gamma/2} y dH(y) .$$

A least upper and a greatest lower bound on $E \epsilon_r$ has been determined so that $E \epsilon_r$ lies in the range

$$-\frac{\gamma}{2} \leq E \epsilon_r \leq \frac{\gamma}{2} . \quad (61)$$

Thus $E R(t)$ is bounded by

$$\mu - \frac{\gamma}{2} \leq E R(t) \leq \mu + \frac{\gamma}{2} . \quad (62)$$

In most of the later work $E R(t)$ will be estimated from measurements and used to estimate μ , which will be unknown a priori. In cases of this sort the inequality above shows that there will be an uncertainty of $\pm \gamma/2$ in determining μ from measured values of $E R(t)$.

The mean square deviation of $R(t)$ from μ has been established as

$$E (R(t) - \mu)^2 = \sigma_{r_1}^2 + \sigma_m^2 + E \epsilon_r^2 \quad (63)$$

$E \epsilon_r^2$ is given by

$$E \epsilon_r^2 = \sum_{n=0}^{N-1} \int_{-\gamma/2}^{\gamma/2} (y^2 + \frac{\gamma^2}{4}) dH(y) .$$

A least upper and a greatest lower bound have been determined for $E \epsilon_r^2$ so that

$$\frac{\gamma^2}{4} \leq E \epsilon_r^2 \leq \frac{\gamma^2}{2} . \quad (64)$$

Thus

$$\sigma_{r_1}^2 + \sigma_m^2 + \frac{\gamma^2}{4} \leq E (R(t) - \mu)^2 \leq \sigma_{r_1}^2 + \sigma_m^2 + \frac{\gamma^2}{2} . \quad (65)$$

$\sigma_{r_1}^2$ is the mean square deviation of the output meter input from μ . Thus the inequalities above show that the use of an output meter, such as described by the chosen mathematical model, to measure the properties of an input $r_1(t)$ will increase the mean square deviation from μ by at least $\sigma_m^2 + \gamma^2/4$ but by no more than $\sigma_m^2 + \gamma^2/2$.

It is important to note that subject to the assumptions stated above the contribution of the output meter to both bias error and mean square error appears as a simple addition to the corresponding errors in the input to the output meter. Thus the effect of the output meter on bias and mean square errors can be considered independently of other components of the measurement system.

Now consider the bias error given by

$$\text{bias error} = E R(t) - \mu = E \epsilon_r . \quad (66)$$

The magnitude of this bias error is bounded by $\gamma/2$. For a fixed meter movement with several sensitivities

$$\epsilon_r = E_r V$$

and bias error is given by

$$\text{bias error} = E E_r V . \quad (67)$$

Equation (67) shows that for fixed E_r bias error is a minimum for minimum V .

This is an appropriate place to consider a related problem which arises in a later chapter. Consider a case where μ is a function of time. To be specific let μ be given by

$$\mu = Af(t)$$

where A is a constant which is being estimated by the measurement system and $f(t) > 0$. It turns out that under certain conditions the bias error is given by

$$\text{bias error} = \frac{V}{f(t)} E_r \quad (68)$$

when μ is the expected value of the input to the output meter. The question to be considered is how to minimize the bias error. If equation (68) were the only restraint it is clear that for E_r constant, the largest possible value of $f(t)$ and the smallest value for V would yield the smallest bias error. However, if $Af(t)$ is the expected value of the input to the output meter, as is being assumed here, equation (68) is not the only restraint. In addition to equation (68) the output meter must operate in its linear range. Thus the definition of linear range requires

$$4\sigma_{r_2} < Af(t) < Ny - 4\sigma_{r_2}$$

where the quantities other than $Af(t)$ will be assumed fixed. It is convenient to denote the full scale reading in output units as F.S. so that Ny is given by

$$Ny = V \text{ F.S.}$$

Now to minimize bias error subject to the restraint imposed by linear range $f(t)$ should be made as large as possible subject to the restraint

$$Af(t) + 4\sigma_{r_2} < V \text{ F.S.} .$$

The latter restraint can be written as

$$\frac{A}{\text{F.S.}} + \frac{4\sigma_{r_2}}{\text{F.S.} \cdot f(t)} < \frac{V}{f(t)} .$$

Examination of this inequality shows that if

$$\frac{4\sigma_{r_2}}{\text{F.S.} \cdot f(t)} < < \frac{A}{\text{F.S.}}$$

adjustment of $f(t)$ for minimum bias error forces $\frac{V}{f(t)}$ to be approximately equal to $\frac{A}{\text{F.S.}}$. Thus if this adjustment is made and if A and F.S. are constant equation (59) shows that bias error is essentially constant and independent of $\frac{V}{f(t)}$.

CHAPTER VII

FORMULATION OF GENERAL EXPRESSIONS FOR MEASUREMENT SYSTEM ERRORS

It is the purpose of this chapter to consider the operation of the linear noisy measurement system as a whole and formulate general expressions for its errors. Several portions of the linear noisy measurement system which perform basically different operations have been identified in the preceding chapters. Since these distinct element groups must be interconnected to form the composite system, several modes of operation are possible. The mathematical model chosen for the biased estimating system admits many variations in this group of elements with no necessity for distinguishing cases. Consideration of the complete system, however, necessitates a choice as to whether the operation to remove bias precedes or follows the output meter. A further variation in the mode of operation is afforded by the choice of the number and timing of the readings of the output meter.

The first objective of this chapter will be to distinguish those cases which warrant separate consideration. Consider first the operation to remove bias. Such an operation must be performed in any measurement system. Conventionally measurement systems are read in the steady-state so that the bias can be removed by the calibration of the output meter. Using the mathematical model for the output meter developed in Chapter VI, it is possible to compare operation of a system with the bias removed before the output meter with a system which removes the bias after the

output meter is read. Expressions for the bias error and the mean square error for these two conditions of operation will now be formulated.

Consider the diagram of Figure 13a representing the case that the bias is removed before the output meter. $E e(t)$ is given by equation (25) as

$$E e(t) = S H_1(t).$$

Thus the box labeled operation to remove bias must divide $e(t)$ by $H_1(t)^*$. Such a division operation could be performed on physical signals by existing analog components.

For this type of operation equations (63) and (66) give the bias and mean square errors as

$$E R_a(t) - S = E e_r$$

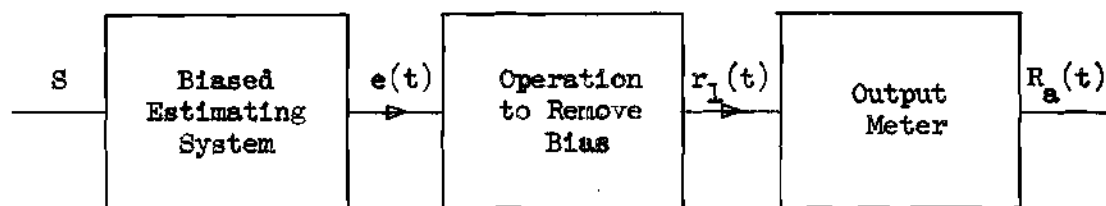
$$E (R_a(t) - S)^2 = \frac{E e(t)^2}{H_1(t)^2} - S^2 + E e_m^2 + E e_r^2. \quad (69)$$

It is convenient to show the meter sensitivity explicitly in these equations. Hence, the above equations can be written as

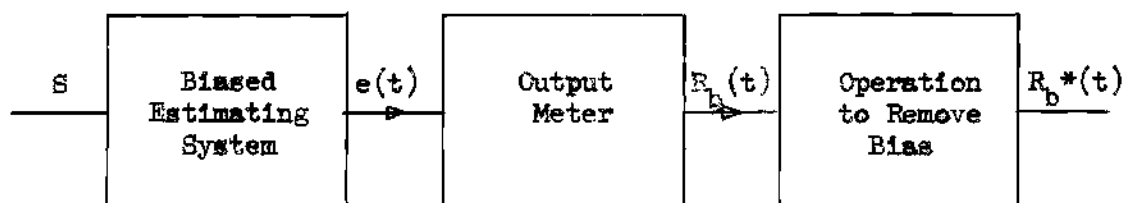
$$E R_a(t) - S = V_a E e_r$$

$$E (R_a(t) - S)^2 = \frac{E e(t)^2}{H_1(t)^2} - S^2 + V_a^2 (E e_m^2 + E e_r^2) \quad (70)$$

*It will be assumed here and in the work to follow that at the time the meter is read $H_1(t) \neq 0$.



(a)



(b)

Figure 13 Two Possible Modes of Measurement System Operation

where these errors are referred to the input to the output meter.

For the system of Figure 13b straightforward use of equations (63) and (66) gives for this case

$$E R_b(t) = \frac{S H_1(t)}{V_b} + E E_r$$

$$E(R_b(t) - S H_1(t))^2 = \frac{E e(t)^2 - S^2 H_1(t)^2}{V_b^2} + E E_m^2 + E E_r^2 \quad (71)$$

where the errors are referred to the measurement system output. Note that the symbol V_b has been used here for meter sensitivity to indicate a possible difference in meter sensitivity between mode (a) and mode (b) operation. Although the bias of $R_b(t)$ cannot be removed completely, division of $R_b(t)$ by $H_1(t)$ will remove the bias produced by system elements other than the output meter. The bias produced by the output meter cannot be removed in most cases.

Thus if $R_b^*(t)$ is given by

$$R_b^*(t) = \frac{R_b(t) V_b}{H_1(t)}$$

the bias and mean square errors of $R_b^*(t)$ can be expressed as

$$E R_b^*(t) - S = \frac{V_b}{H_1(t)} E E_r$$

$$E (R_b^*(t) - S)^2 = \frac{E e(t)^2}{H_1(t)^2} - S^2 + \frac{V_b^2}{H_1(t)^2} [E E_m^2 + E E_r^2] \quad (72)$$

where the errors are referred to the output of the measurement system. Examination of equations (70) and (72) shows that these two equations would be equivalent if the meter sensitivity V_a were given by

$$V_a(t) = \frac{V_b}{H_1(t)} \quad (73)$$

Thus mode (a) and mode (b) operation are essentially equivalent. To be specific the bias and mean square errors calculated for mode (a) referred to the input of the output meter are the same as the corresponding errors of mode (b) referred to the output of the output meter if V_a is given by equation (73). Thus any major difference between mode (a) and mode (b) operation if the sensitivities are related as given by equation (73) is dictated by practical and not theoretical considerations.

Several of the practical considerations which would influence a choice between mode (a) and mode (b) operation are the following: (1) The range of sensitivities which are possible with a fixed meter movement* is limited, thus it may not be possible to achieve the sensitivity $V_a(t_1)$ required for the (a) and (b) modes to be equivalent in a particular case. (2) For mode (b) operation the linear range must be determined from $SH_1(t)$ instead of simply S . Thus operation in the linear range requires

$$4\sigma_{r2}^2 < \min_{[0,t]} SH_1(t) < \max_{[0,t]} SH_1(t) < N\gamma - 4\sigma_{r2}^2 \quad (74)$$

Difficulty is encountered if $SH_1(t) < 0$ for some but not all t . In this case $SH_1(t)$ is sometimes outside the linear range as previously defined.

* This term is used here to refer to any read-out device, and is not restricted to an electrical D'Arsonval meter movement.

This difficulty can be circumvented in several ways, for example, by modifying the definition of linear range or by adding a fixed constant to $e(t)$. The details of such expedients do not contribute to the general discussion; hence suffice it to say that the required linear range is greater in most cases for mode (b) operation. (3) The operation to remove bias can be performed on paper for mode (b) operation while for mode (a) operation it must be performed with analog computing components. Thus there is a difference in the amount of noise and uncertainty added by the operation to remove bias in the two cases. (4) The effective meter sensitivity and hence the calibration of the output meter is time variable for mode (b) operation, while it is a time independent constant for mode (a) operation. (5) Equation (72) shows that the bias error for mode (b) is a function of $H_1(t)$. Thus, as discussed in Chapter VI, bias error can be minimized by adjusting the system gain so that the input to the output meter is as large as possible but still within the linear range of the meter. If such an adjustment is made before every reading, bias error will be essentially independent of $V_b/H_1(t)$. In almost all cases of importance the gain adjustment can be made without affecting the mean square error.

Expressions for the mean and correlation functions of the output of the output meter will now be formulated for the connection of Figure 13a which is the logical theoretical, if not practical, choice. The output meter sensitivity will be treated as a time variable so that the results of the analysis will apply to either the (a) or (b) mode of operation. The operation to remove bias is considered to be an ideal noiseless operation since in most practical cases this operation will be

performed "on paper" after the output meter. Since it is necessary to consider only one case the notation $R(t)$ for the system output and $V(t)$ for the meter sensitivity will be adopted.

Since the connection of Figure 13b is most likely to be the one used in practice an explicit statement of the way in which the results of the analysis apply to this case will now be made. All results pertaining to $R(t)$ referred to the input of the output meter will apply to the output of the system of Figure 13b (denoted by $R_b^*(t)$), if the meter sensitivity $V(t)$ is given by

$$V(t) = \frac{V_b}{H_1(t)}$$

where V_b is the actual output meter sensitivity and $H_1(t)$ is the appropriate step response for the system. This statement is of course subject to the qualification that there is no practical restriction on the mode (b) operation. For example, the linear range of the output meter must accommodate $SH_1(t)$. For mode (b) operation the fact that the effective meter sensitivity changes with time causes the meter deflection corresponding to a fixed input to vary with time. Of course it reaches a fixed value in the steady-state.

To proceed with the analysis, assume the mode (a) connection but treat the meter sensitivity as a time variable. Thus $r_1(t)$ is given by

$$r_1(t) = \frac{e(t)}{H_1(t)} \quad , \quad (75)$$

and the expected value of $r_1(t)$ by

$$E r_1(t) = S . \quad (76)$$

The correlation function of $r_1(t)$ is given by

$$E r_1(t_1)r_1(t_2) = \frac{E e(t_1)e(t_2)}{H_1(t_1)H_1(t_2)} = \frac{\phi(t_1, t_2)}{H_1(t_1)H_1(t_2)} \quad (77)$$

where $\phi(t_1, t_2)$ is given by equation (33).

The output of the output meter, $R(t)$, is given by equation (57) as

$$R(t) = r_1(t) + V(t) (E_m + E_r) \quad (78)$$

while the bias and mean square errors are given by

$$E R(t) - S = V(t) E E_r \quad (79)$$

$$E (R(t) - S)^2 = E r_1(t)^2 - S^2 + V(t)^2 (E E_m^2 + E E_r^2) . \quad (80)$$

Note that as mentioned in Chapter VI the errors due to the biased estimating system and due to the output meter can be identified and separated in the general expressions. These separate errors can be listed as follows:

output meter errors

$$\text{bias error} = V(t) E E_r$$

$$\text{mean square error} = V(t)^2 [E E_m^2 + E E_r^2] \quad (81)$$

biased estimating system errors

$$\text{bias error} = 0$$

$$\text{mean square error} = E r_1(t)^2 - S^2 = \sigma_{r_1}^2 . \quad (82)$$

Now consider minimizing the total bias and mean square errors.

Bias error is more important to measurement system behavior than mean square error, hence bias error will be minimized first, after which mean square error will be minimized subject to the restraint which minimized bias error. With the true value of the quantity to be measured, S , unknown the round-off error has equal probability of being positive or negative. Thus in a sense* the total bias error is minimized by removing the bias error from the output of the biased estimating system. In almost all cases of interest the bias error due to the biased estimating system can be removed and the resulting overall bias error can be minimized by varying system gains which do not affect the mean square error. Thus the mean square error can be minimized independently of bias error. Since the component mean square errors due to the biased estimating system and output meter are both positive and independent of each other, the total mean square error can be minimized by minimizing the component errors separately.

Now consider several readings of the output meter at times t_1, t_2, \dots, t_n , and an estimate L_n of S formed as a linear function of the readings $r(t_1), \dots, R(t_n)$. L_n can be expressed as

$$L_n = \frac{\sum_{i=1}^n A_i R(t_i)}{\sum_{i=1}^n A_i} \quad (83)$$

* To be specific, if S is regarded as a random variable uniformly distributed over the linear range of the output meter, then $E E_r = 0$ independent of $V(t)$. In such a case the total bias error for a number of readings is minimized by removing the bias from the output of the biased estimating system.

where the A_i are arbitrary constants. $E L_n$ is easily found to be

$$E L_n = S + \frac{\sum_{i=1}^n A_i V(t_i) E E_r(t_i)}{\sum_{i=1}^n A_i} .$$

$E E_r(t_i)$ has been assumed to be independent of t_i . Thus the bias error of L_n can be expressed as

$$E L_n - S = \frac{E E_r \sum_{i=1}^n A_i V(t_i)}{\sum_{i=1}^n A_i} . \quad (84)$$

Note that $\frac{\sum_{i=1}^n A_i V(t_i)}{\sum_{i=1}^n A_i}$ is a weighted average of the sensitivities of

the output meter at the times t_1, t_2, \dots, t_n .

The mean square error of L_n can be expressed as

$$E (L_n - S)^2 = \frac{\sum_{i=1}^n \sum_{j=1}^n A_i A_j E [R(t_i) - S] [R(t_j) - S]}{\left[\sum_{i=1}^n A_i \right]^2} \quad (85)$$

Using the expression of equation (78) for $R(t)$, equation (85) can be written as

$$\begin{aligned}
E (L_n - S)^2 &= \left[\sum_{i=1}^n A_i \right]^{-2} \left[\sum_{i=1}^n \sum_{j=1}^n A_i A_j E [r_1(t_i) - S] (r_1(t_j) - S) \right. \\
&\quad \left. + \sum_{i=1}^n \sum_{j=1}^n A_i A_j E \epsilon_r(t_i) \epsilon_r(t_j) + E \epsilon_m^2 \sum_{i=1}^n A_i^2 \right] .
\end{aligned} \tag{86}$$

Equation (86) shows the effect of the output meter separately from the effect of the biased estimating system. Thus it seems reasonable to define

L_n' as

$$L_n' = \frac{\sum_{i=1}^n A_i r_1(t_i)}{\sum_{i=1}^n A_i} \tag{87}$$

and identify the first term in equation (86) as the mean square error of L_n' . It should be noted that L_n' depends only on the biased estimating system and not on the output meter.

It seems reasonable to assume that in most practical cases $\epsilon_r(t_i)$ will be independent of $\epsilon_r(t_j)$ for practical differences in t_i and t_j .

If this assumption is made

$$\begin{aligned}
\sum_{i=1}^n \sum_{j=1}^n A_i A_j E \epsilon_r(t_i) \epsilon_r(t_j) &= E \epsilon_r^2 \sum_{i=1}^n A_i^2 \\
&+ E \epsilon_r^2 \sum_{i=1}^n \sum_{\substack{j=1 \\ j \neq i}}^n A_i A_j .
\end{aligned} \tag{88}$$

Now by use of equation (86) and (87) the total mean square error of L_n can be expressed as

$$E (L_n - S)^2 = E (L_n' - S)^2 + \frac{E \epsilon_r^2 + E \epsilon_{in}^2 \sum_{i=1}^n A_i^2 + E \epsilon_r^2 \left[\left(\sum_{i=1}^n A_i \right)^2 - \sum_{i=1}^n A_i^2 \right]}{\left[\sum_{i=1}^n A_i \right]^2} \quad (89)$$

with $E (L_n' - S)^2$ given by

$$E (L_n' - S)^2 = \frac{\sum_{i=1}^n \sum_{j=1}^n A_i A_j E \left[r_1(t_i) - S \right] \left[r_1(t_j) - S \right]}{\left[\sum_{i=1}^n A_i \right]^2} \quad (90)$$

It can be shown easily that $E \left[r_1(t_i) - S \right] \left[r_1(t_j) - S \right]$ is given by

$$E \left[r_1(t_i) - S \right] \left[r_1(t_j) - S \right] = \frac{\phi(t_i, t_j)}{H_1(t_i) H_1(t_j)} - S^2 \quad (91)$$

This quantity will be denoted $\lambda(t_i, t_j)$. Since

$$\phi(t_i, t_j) = \phi(t_j, t_i)$$

it is possible to order the arguments of ϕ so that

$$t_j \geq t_i.$$

Thus equation (33) can be used to express $\lambda(t_i, t_j)$ as

$$\lambda(t_1, t_j) = \frac{\pi S}{H_1(t_1)H_1(t_j)} \sum_{q=1}^J \left[SA_1^2 \int_0^{t_1} h_q(u)h_q(u+t_j-t_1)|K_q(t_1-u)|du \right. \\ \left. + B_1^2 \int_0^{\infty} h_q(u)h_q(u+t_j-t_1)du \right], \quad t_j \geq t_1. \quad (92)$$

Equation (90) can be written in terms of $\lambda(t_1, t_j)$ as

$$E(L_n' - S)^2 = \frac{\sum_{i=1}^n \sum_{j=1}^n A_i A_j \lambda(t_i, t_j)}{\left[\sum_{i=1}^n A_i \right]^2} \quad (93)$$

In the special case that $n=1$ equations (84) and (89) reduce to equations (79) and (80) respectively.

A comparison of equations (79) and (84) shows that the bias error caused by the output meter is affected by the number of readings only to the extent that the weighted average

$$\frac{\sum_{i=1}^n A_i V(t_i)}{\sum_{i=1}^n A_i} \quad \text{differs from } V(t).$$

Thus if changes in meter sensitivity are disregarded, the bias error caused by the output meter is independent of the number of readings of the output meter.

On the other hand equation (89) shows that the mean square error caused by the output meter in n readings can be expressed as

$$E \epsilon_m^2 \frac{\sum_{i=1}^n A_i^2}{\left[\sum_{i=1}^n A_i \right]^2} + \left[E \epsilon_r^2 \right]^2 + \left[E \epsilon_r^2 - \left[E \epsilon_r^2 \right]^2 \right] \frac{\sum_{i=1}^n A_i^2}{\left[\sum_{i=1}^n A_i \right]^2} .$$

It can easily be shown* that

$$\frac{\sum_{i=1}^n A_i^2}{\left[\sum_{i=1}^n A_i \right]^2} \geq \frac{1}{n}$$

Thus, since all terms in the above expression are positive, the mean square error caused by the output meter is bounded below by

$$\frac{E \epsilon_m^2}{n} + \left[E \epsilon_r^2 \right]^2 + \frac{E \epsilon_r^2 - \left[E \epsilon_r^2 \right]^2}{n}$$

a bound which is realized for example if $A_i = 1$ for all i . Comparison of this expression to the mean square error due to the output meter in a single reading which can be inferred from equation (80), shows that this error is clearly reduced as n increases from one. It is minimized with respect to the A_i if these constants are chosen so that

$$\sum_{i=1}^n A_i^2 - \frac{1}{n} \left[\sum_{i=1}^n A_i \right]^2 = 0 . \quad (94)$$

*See for example Hardy and Littlewood (21).

Similar statements giving the general effect of more than one reading on the errors due to the biased estimating system are more difficult to make. If it is assumed that the bias is removed from the output of the biased estimating system in the same fashion that it was for a single reading, then the bias error for more than one reading is still zero. However, the mean square error, given by equation (90) or (93) is more difficult to interpret. One special case does give tractable results and comments will be restricted to this case.

Consider the case that the A_i are all one and the $\lambda(t_i, t_j)$ are zero if t_i is not equal to t_j . For this case equation (93) becomes

$$E (L_n' - S)^2 = \frac{\sum_{i=1}^n \lambda(t_i, t_i)}{n^2} .$$

If the $\lambda(t_i, t_i)$ are all equal the equation reduces further to

$$E (L_n' - S)^2 = \frac{\lambda(t, t)}{n} = \frac{\sigma_{r1}^2(t)}{n} .$$

This last equation shows that use of n reading in this special case reduces the mean square error due to the output meter by dividing the mean square error of a single reading by n .

The problem of choosing the constants A_i in a linear estimate, L_n , so as to minimize the overall mean square error, as well as certain other minimization problems, is beyond the scope of the present program. Without the solution to such minimization problems it is difficult to make further general statements regarding the use of more than one reading of the output

meter. Before leaving this topic, however, examination of the simple case of $n = 2$ will serve to give further insight into the effect of several readings.

In the case of $n = 2$ the restraint imposed by equation (94) requires that $A_1 = A_2$ so that L_2 is given by

$$L_2 = \frac{1}{2} \sum_{i=1}^2 R(t_i)$$

and L_2' by

$$L_2' = \frac{1}{2} \sum_{i=1}^2 r_1(t_i) \quad .$$

The bias error of L_2 is given by equation (84) as

$$E L_2 - S = E E_r \frac{1}{2} [V(t_1) + V(t_2)]$$

while the mean square error of L_2 is given by equation (89) as

$$E (L_2 - S)^2 = E (L_2' - S)^2 + \frac{E \epsilon_r^2}{2} + \frac{E \epsilon_m^2}{2} + \frac{(E \epsilon_r)^2}{2} \quad .$$

Equation (93) yields for $E (L_n' - S)^2$ the expression

$$E (L_n' - S)^2 = \frac{\lambda(t_1, t_1) + 2\lambda(t_1, t_2) + \lambda(t_2, t_2)}{4} \quad .$$

The $\lambda(t_i, t_j)$ are given explicitly in terms of the properties of the biased system by equation (92).

The restriction imposed on the $\lambda(t_i, t_j)$ by the fact that they are moments of a random variable can be stated in this case as

$$\lambda(t_1, t_1) > 0; \lambda(t_2, t_2) > 0$$

$$\lambda(t_1, t_1) \lambda(t_2, t_2) - \lambda(t_1, t_2)^2 > 0 .$$

Thus if $\lambda(t_1, t_2)$ was not restricted by the properties of the biased estimating system, it would be bounded by

$$-\sqrt{\lambda(t_1, t_1) \lambda(t_2, t_2)} < \lambda(t_1, t_2) < \sqrt{\lambda(t_1, t_1) \lambda(t_2, t_2)}$$

so that $E (L_n' - S)^2$ would be bounded by

$$\frac{1}{4} \left[\lambda(t_1, t_1) + \lambda(t_2, t_2) - 2 \sqrt{\lambda(t_1, t_1) \lambda(t_2, t_2)} \right] < E (L_n' - S)^2$$

and

$$\frac{1}{4} \left[\lambda(t_1, t_1) + \lambda(t_2, t_2) + 2 \sqrt{\lambda(t_1, t_1) \lambda(t_2, t_2)} \right] > E (L_n' - S)^2 .$$

If $\lambda(t_1, t_2)$ were zero, as would be the case if $r_1(t_1)$ were independent of $r_1(t_2)$, then $E (L_n' - S)^2$ is given by

$$E (L_n' - S)^2 = \frac{\lambda(t_1, t_1) + \lambda(t_2, t_2)}{4}$$

It is significant to note that $\lambda(t_1, t_2)$ can be negative so that the case of $\lambda(t_1, t_2) = 0$ does not result in the minimum value of $E (L_n' - S)^2$.

Although the bounds on $\lambda(t_1, t_2)$ given above are modified by the properties of the biased estimating system, it is possible to choose elements of this system so that $\lambda(t_1, t_2)$ is negative.

CHAPTER VIII

BIASED ESTIMATING SYSTEM ERRORS IN CERTAIN SPECIAL CASES

The mathematical model for the output meter formulated in Chapter VI was such that if the round-off unit γ was smaller than the root mean square error of the biased estimating system then the effect of the output meter on the overall bias and mean square errors was independent of the bias and mean square errors of the biased estimating system. The expressions for more general cases obtained in Chapter VII* bear out this fact and indicate that for a single reading of the output meter and in other cases of importance the output meter errors can be treated independently of the biased estimating system errors. The expressions given in Chapter VII for the errors due to the output meter are relatively simple and easy to interpret, while those for the biased estimating system are more complicated.

Thus the purpose of this chapter is to consider certain special cases for which the expressions for the biased estimating system errors are simplified. Specifically, the discussion of this chapter will be limited to the case of a single reading of the output, $r_1(t)$, of the operation to remove bias. The effect of using more than one reading to form an estimate of S has been considered in Chapter VII. Since the bias error of $r_1(t)$ is zero, attention can be restricted to the mean square error.

The mean square error of a single reading of the output of the

* See for example equations (84) and (89).

biased estimating system with the bias removed can be expressed through use of equation (35) as

$$E (R_1(t) - S)^2 = \frac{\pi}{H_1(t)^2} \sum_{i=1}^J \left[SA_1^2 \int_0^t h_1(u)^2 |K_1(t-u)| du + B_1^2 \int_0^\infty h_1(u)^2 du \right]. \quad (95)$$

Considerable insight into the dependence of this mean square error on the nature of the system elements can be achieved by explicitly indicating the gain and predominant characteristic frequency of each element group in the expression for mean square error. This can be accomplished by a normalization procedure.

First, consider the element group gain. It has been assumed that the step response of each element group reaches a limiting finite non-zero value for large values of time. This limiting value will be taken as the element group gain, $T(i,i)$. Thus

$$\lim_{t \rightarrow \infty} G_1(t) = T(1,i). \quad (96)$$

In general the notation $T(u,v)$ will be adopted for the gain between the points u and v in a system. Thus, for example,

$$\lim_{t \rightarrow \infty} H_1(t) = T(1,j). \quad (97)$$

and

$$\lim_{t \rightarrow \infty} K_1(t) = T(1,1-1)^* \quad (98)$$

* $T(1,0) = 1$.

where J is the total number of element groups in the system (refer to Figure 9). Normalization can now be carried out with respect to the gain of each element group. Thus if $g_1'(t)$ is the normalized impulse response of the i^{th} element group

$$g_1'(t) = \frac{g_1(t)}{T(1,1)} .$$

Equation (95) can be expressed in a straightforward way in terms of the normalized responses as

$$E [r_1(t) - s]^2 = \frac{\kappa}{H_1'(t)^2} \sum_{i=1}^J \frac{1}{T(1,i-1)} \left[SA_1^2 \int_0^t h_1'(u)^2 |K_1'(t-u)| du \right. \\ \left. + \frac{B_1^2}{T(1,i-1)} \int_0^\infty h_1'(u)^2 du \right] . \quad (99)$$

Equation (99) will be further normalized with respect to the frequency dependence of the elements. Since it is not practical to normalize each element group transfer function with respect to a different characteristic frequency, the procedure to be followed will be to normalize all of its element groups with respect to an arbitrarily selected characteristic frequency. This frequency will be denoted ω_c . The effect of this frequency normalization will be to express, for example, the mean square error in terms of the characteristics of prototype element groups which are normalized with respect to ω_c . Thus the effect of changing the characteristic frequency by scaling the element values for a fixed type of system is placed

explicitly in evidence. In addition to the theoretical insight gained by the normalization, it will also facilitate the direct use of prototype response curves for system element groups.

The normalization can be accomplished by a change of variable. In the frequency domain a variable, for example, $\bar{g}_1''(s)$, is introduced so that

$$g_1''\left(\frac{s}{\omega_c}\right) = \bar{g}_1''(s) \quad . \quad (100)$$

Note in particular that

$$\bar{g}_1''(j_1) = g_1(j\omega_c) \quad .$$

Now consider the inverse Laplace transforms of $\bar{g}_1(s)$ and $\bar{g}_1''(s)$. It follows immediately from equation (100) that

$$L^{-1} \left[\bar{g}_1(s) \right] = g_1(t) = L^{-1} \left[\bar{g}_1''\left(\frac{s}{\omega_c}\right) \right] \quad .$$

If the notation

$$L^{-1} \left[\bar{g}_1''(\dot{x}) \right] = g_1''(t)$$

is adopted, reference to a table of Laplace transform pairs* gives for $g_1(t)$

$$g_1(t) = \omega_c g_1''(t\omega_c) \quad . \quad (101)$$

* See for example Gardner and Barnes (16).

Straightforward computation yields

$$G_1(t) = \int_0^t g_1(u) du = \int_0^{\omega_c t} g_1''(u) du = G_1''(\omega_c t) \quad (102)$$

and

$$\int_0^t g_1(u)^2 du = \omega_c \int_0^{\omega_c t} g_1''(u)^2 du \quad (103)$$

Similar relations will exist between the $h_1(t)$ and the $h_1''(t)$ and between the $k_1(t)$ and the $k_1''(t)$. If equation (95) is thus normalized with respect to frequency there results

$$\begin{aligned} E(r_1 - s)^2 = & \frac{\pi \omega_c}{H_1''(\omega_c t)^2} \sum_{i=1}^J \left[S A_1^2 \int_0^{\omega_c t} h_1''(u)^2 |K_1''(\omega_c t - u)| du \right. \\ & \left. + B_1^2 \int_0^{\infty} h_1''(u)^2 du \right] . \end{aligned} \quad (104)$$

The notation $g_1^*(t)$, for example, will be adopted for an element group variable which is normalized with respect to both amplitude and frequency. Thus the desired expression for mean square error normalized with respect to both amplitude and frequency is

$$E (r_1 - S)^2 = \frac{\pi \omega_c}{H_1^*(\omega_c t)^2} \sum_{i=1}^J \frac{1}{T(1, i-1)} \left[SA_1^2 \int_0^{\omega_c t} h_1^*(u)^2 |K_1^*(\omega_c t - u)| du \right. \\ \left. + \frac{B_1^2}{T(1, i-1)} \int_0^{\infty} h_1^*(u)^2 du \right] . \quad (105)$$

Examination of equation (105) shows explicitly the effect of the element group gains and the lower characteristic frequency ω_c .

The nature of the element groups of a measurement system to measure a constant is necessarily such that each element group is in effect a low-pass filter. Two important characteristics of a typical such filter are its gain and cut-off frequency. These two numbers are adequate to specify a particular filter made from a given prototype. The prototype filter itself, however, is more difficult to characterize. Its behavior depends on the number and nature of its elements and their relative values. If the type filter is fixed, for example, a Butterworth filter, families of prototype curves which show the effect of different relations between the element values are available in the literature*.

Returning to equation (105), the effect of element gain and ω_c is immediately apparent. Thus further consideration can be restricted to the "prototype" behavior of the element groups.

Before proceeding, however, it is worth while to write down the obvious limiting expressions for the mean square error which result

* See for example K. W. Henderson and W. H. Kantz (22).

from extreme values of ω_c and the $T(u, v)$.

First of all it is clear that for fixed $\omega_c t$, $E (r_1 - S)^2$ approaches zero with ω_c . Thus in designing a measurement system ω_c should be made as small as possible.

Since the upper limit of integration of the first term of the summand of equation (105) is $\omega_c t$ and its lower limit is zero, there will be systems with a value of ω_c so small that the first integral is negligible with respect to the second. In this case

$$E (r_1 - S)^2 \approx \frac{\pi \omega_c}{H_1^*(\omega_c t)^2} \sum_{i=1}^J \frac{B_i^2}{T(1, i-1)^2} \int_0^{\infty} h_i^{*2}(u) du \quad (106)$$

where $f(x) \approx g(x)$ denotes that $f(x)$ is approximately equal to $g(x)$.

Now consider the effect of the gains $T(u, v)$. For this purpose it is expedient to write equation (105) as

$$E (r_1(t) - S)^2 = \frac{\pi \omega_c}{H_1^*(\omega_c t)^2} \left[SA_1^2 \int_0^{\omega_c t} h_1^*(u)^2 du + B_1^2 \int_0^{\infty} h_1^*(u)^2 du \right] + \epsilon \quad (107)$$

where, assuming $T(u, v) \geq 1$ all u and v ,

$$\epsilon \leq \frac{1}{T(1, 1)} \frac{\pi \omega_c}{H_1^*(\omega_c t)^2} \sum_{i=2}^J \left[SA_i^2 \int_0^{\omega_c t} h_i^*(u)^2 |K_i^*(\omega_c t - u)| du + \frac{B_i^2}{T(1, i-1)} \int_0^{\infty} h_i^{*2}(u) du \right] .$$

Examination of the expression for ϵ shows that ϵ approaches zero as the gain of the first element group $T(1,1)$ approaches infinity. Thus there are cases of importance for which $T(1,1)$ is so large that

$$E [r_1(t) - S]^2 \approx \frac{\pi \omega_c}{H_1^*(\omega_c t)^2} \left[SA_1^2 \int_0^{\omega_c t} h_1^*(u)^2 du + B_1^2 \int_0^{\infty} h_1^*(u)^2 du \right] \quad (108)$$

In this case use of the Schwarz inequality and the fact that

$$\int_0^{\omega_c t} h_1^*(u)^2 du \leq \int_0^{\infty} h_1^*(u)^2 du$$

yields

$$E [r_1(t) - S]^2 \geq \frac{\pi}{t} (SA_1^2 + B_1^2) \quad (109)$$

This equation is significant because it is not only a lower bound on the mean square error for the case of large $T(1,1)$, but also a lower bound on the mean square error in general since the ϵ of equation (107) is always positive.

A upper bound on $E [r_1(t) - S]^2$ for the case that $T(1,1)$ is very large can be obtained from equation (108) as

$$E [r_1(t) - S]^2 \leq \pi \omega_c (SA_1^2 + B_1^2) \frac{\int_0^{\infty} h_1^*(u)^2 du}{H_1^*(\omega_c t)^2}$$

by noting that

$$\int_0^{\omega_c t} h_1^*(u)^2 du \leq \int_0^{\infty} h_1^*(u)^2 du .$$

For large t this bound approaches

$$\pi \omega_c (SA_1^2 + B_1^2) \int_0^{\infty} h_1^*(u)^2 du$$

since $H_1^*(\infty) = 1$.

For large $T(1,1)$ and small ω_c it is clear that either equation (106) or equation (108) reduces to

$$E [r_1(t) - S]^2 \approx \frac{\pi \omega_c}{H_1^*(\omega_c t)^2} B_1^2 \int_0^{\infty} h_1^*(u)^2 du . \quad (110)$$

If B_1^2 should be zero or much less than SA_1^2 , equation (109) reduces to

$$E [r_1(t) - S]^2 \geq \frac{\pi}{t} SA_1^2 . \quad (111)$$

In this case $\pi SA_1^2/t$ is a greatest lower bound. This fact can be proved by choosing

$$h_1^*(u) = \frac{1}{\omega_c} e^{-\frac{\alpha}{\omega_c} u} ; u \geq 0 .$$

For this $h_1^*(u)$, $E [r_1(t) - S]^2$ is given by

$$E [r_1(t) - S]^2 = \frac{\omega_c \int_0^{\omega_c t} e^{-\frac{2\alpha}{\omega_c} u} du}{\left[\int_0^{\omega_c t} e^{-\frac{\alpha}{\omega_c} u} du \right]^2} \pi SA_1^2 .$$

If α is allowed to approach zero, $E[r_1(t) - S]^2$ becomes in the limit

$$E(r_1(t) - S)^2 = \frac{\pi}{t} SA_1^2,$$

showing that $\pi SA_1^2/t$ is a greatest lower bound for the class being considered.

Now return to the general expression for mean square error given by equation (105). As pointed out above, normalization of the impulse and step response characteristics of the element groups has made it possible to consider the effect of element group gain and characteristic frequency independently of what has been termed the "prototype" behavior. The prototype behavior is obtained by setting ω_c and all of the gain terms equal to one in equation (105). This yields

$$E[r_1(t) - S]^2 \text{ (prototype)} = \frac{\pi}{H_1^*(t)^2} \sum_{i=1}^J \left[SA_i^2 \int_0^t h_i^*(u)^2 |K_i^*(t-u)| du \right. \\ \left. + B_i^2 \int_0^\infty h_i^*(u)^2 du \right]. \quad (112)$$

CHAPTER IX

SUMMARY AND ENGINEERING APPLICATION OF RESULTS

In the foregoing chapters a mathematical model for a linear noisy measurement system^{*} to measure a constant quantity has been developed and analyzed. The purpose of this chapter will be to summarize the results of the earlier chapters and to emphasize the aspects of the mathematical model which can be useful in the design of measurement systems.

The nature of linear noisy measurement system errors is discussed in Chapter III where the conclusion is reached that in general only the statistical parameters of the output of such a measurement system can be specified.

Bias error and mean square error as defined by equations (2) and (3) of Chapter III are chosen as being the two errors which together come nearest to describing the behavior of the linear noisy measurement system. The bias error of the output of a measurement system gives the amount by which the expected value of the measurement system output differs from the true value of the quantity being measured. The expected value of the output of a measurement system cannot be precisely determined from measurements. However, if the mean square error, σ^2 , is known use can be made of the Bienayme - Tchelysheff inequality given by equation (1) (or in special cases a stronger statement of the same sort) to make statements such as "the probability that a single reading of the measurement

^{*}The definition and restrictions on this type of system are given in Chapter IV.

system differs in absolute value from its expected value by more than $k\sigma$ is less than $1/k^2$ where k is a constant. Thus a good measurement system would be one with a very small, or zero, bias error and a small mean square error. For such a system, with high probability a single reading of the output is very close to the value of the quantity being measured. The major design objective of a linear noisy measurement system will be to minimize bias and mean square errors.

The development above has led to the conclusion that the most general measurement system of the type being considered has three distinct operational parts, namely (1) the biased estimating system, (2) the operation to remove bias and (3) the output meter. These operational parts have distinct properties and distinct errors. The nature of these operational parts of the system is such that the quantity to be measured is applied to the biased estimating system. The output of the biased estimating system can be applied to either the operation to remove bias or the output meter. The differences in these two modes of operation are discussed in Chapter VII where the conclusion is drawn that the two modes of operation are substantially equivalent from a theoretical point of view. However, the practical differences listed in Chapter VII are important. Perhaps the most important consideration in this connection is the fact that if the operation to remove bias follows the output meter then this operation can be performed on paper or by a suitable calibration of the output meter scale. The latter calibration would be of necessity time variable if the measurement system is to be used in the transient condition. In most cases the errors for performance of the operation in this way would be much smaller than the alternative of performing the operation to remove bias with analog components before the output meter. Thus

it is important to note the restrictions that this connection implies, namely (1) the linear range of the output meter as defined in Chapter VI must be adequate to handle the output of the biased estimating system, which can be much larger than the value of the constant to be measured plus the biased estimating system noise; and (2) the effective meter sensitivity, $V(t)$, is given by

$$V(t) = \frac{\text{Actual meter sensitivity}}{H_1(t)}$$

where $H_1(t)$ is the step response from the biased estimating system input to its output.

If a measurement system is designed so that the operation to remove bias follows the output meter, it must be designed with an adequate linear range as discussed above, and $V(t)$ in the general expressions must be determined by the equation above. For this case the errors introduced by the operation to remove bias are so small relative to other errors that they can be neglected.

Assuming that errors due to the operation to remove bias can be neglected, the measurement system errors are those due to the biased estimating system and output meter. The physical assumption that the round-off unit is smaller than the standard deviation of the output of the biased estimating system, in conjunction with the mathematical model, lead to the conclusion that the errors due to the output meter do not depend substantially on the output of the biased estimating system if the output meter is in its linear range. This fact makes it reasonable in a practical case to consider the design of the output meter and biased estimating systems independently.

As is discussed in Chapters VI and VII if the quantity being measured is unknown, as it is in most cases of importance, then the round-off error has equal probability of being positive or negative independent of meter sensitivity. This being the case, in a practical sense, total bias error is minimized by removing the bias from the output of the biased estimating system. The fact that in general the expected value of the output of the biased estimating system is $SH_1(t)$ is sufficient to show that this bias can always be removed by division by $H_1(t)$.

For a single reading of the output meter the fact that the mean square error due to the output meter does not depend on the output of the biased estimating system and the fact that both are positive are sufficient to insure the fact that minimum overall mean square error is achieved by minimizing this type of error for the biased estimating system and the output meter separately.

The situation is more complicated if several readings are made of the output meter and an estimate of the quantity to be measured is formed from some function of these readings. The general problem of choosing an optimum functional form for the estimate and minimizing the mean square error of the estimate by appropriate choice of the parameters of the chosen function and of the biased estimating system is not considered. However, some insight into the effect on the various system errors of more than one reading of the output meter is given in Chapter VII by the consideration of several less general problems.

General expressions are given in equations (84) and (89), for the bias and mean square errors of an estimate, L_n , formed as a linear combination of n reading of the output meter. These expressions are

based on the fact that the bias is removed from the output of the biased estimating system. Equation (89) for the mean square error of L_n shows that this error can be separated into the part due to the biased estimating system and the part due to the output meter. These two parts of the mean square error both depend on the parameters of L_n , but are otherwise independent.

Now turn to a discussion of the mathematical models for the output meter and the biased estimating system. The mathematical model developed for the output meter in Chapter VI accounts for two types of output meter errors namely, movement error and round-off error. Round-off error refers to the error arising because of a non-zero usable scale division, while movement error refers to any other type of output meter error which causes the output meter reading to differ from the true input.

The output meter model has three parameters, the mean square movement error, σ_m , the meter sensitivity, $V(t)$ and the smallest detectable scale division, γ . The bias error due to the output meter for a single reading is given by equation (66) as $E \epsilon_r$. Without knowledge of the true value of the constant being measured it is only possible to bound this error. Equation (64) shows that the bias error due to the output meter is bounded between $\pm \frac{\gamma}{2}$ ^{*}. The effect of output meter sensitivity on this error is discussed in Chapter VI.

The mean square error due to the output meter for a single reading is given by equation (63) as

* γ is the round-off error referred to the input to the output meter. Thus it is expressed in the units of the input to the output meter.

$$\sigma_m^2 + E \epsilon_r^2.$$

$E \epsilon_r^2$ is shown to lie between $\frac{\gamma^2}{4}$ and $\frac{\gamma^2}{2}$. Thus the mean square error due to the output meter for a single reading lies between $\sigma_m^2 + \frac{\gamma^2}{4}$ and $\sigma_m^2 + \frac{\gamma^2}{2}$.

For n readings of the output meter it is shown in Chapter VII that the bias error is affected only in that the weighted average of the sensitivities at the instants of the several readings replaces $V(t)$ in the expression for bias error in terms of the primitive round-off unit E_r . (See for example equations (79) and (84)). On the other hand the mean square error due to the output meter is decreased by using more than one reading. For example if the parameters of a linear estimate L_n are restrained so as to minimize the mean square error due to the output meter, then this portion of the mean square error of L_n is given by

$$\frac{\sigma_m^2}{n} + (E \epsilon_r)^2 + \frac{E \epsilon_r^2 - (E \epsilon_r)^2}{n}.$$

Note that the mean square movement error is $1/n$ th that for a single reading while the mean square round-off error is reduced but by a smaller amount. A restraint on the parameters of a linear estimate which is sufficient to minimize the component of mean square error due to the output meter is given by equation (94).

Now consider the biased estimating system. This operational part of the measurement system is composed of linear elements which can be interconnected in a variety of ways depending on the nature of a particular measurement system. This part of the system is subject to noise,

such as shot noise, introduced by its components. The discussion of Chapter IV shows that the diagram of Figure 4 applies to any biased estimating system for a linear noisy measurement system. This diagram is a representation of the biased estimating system as the tandem connection of several element groups which connect between the points of application of the noise sources. This representation was chosen because the noise sources are unchanged from the physical system. This representation is used in all calculations involving the biased estimating system.

The parameters of the mathematical model for the biased estimating system are the noise per unit bandwidth constants of the noise sources, and the step and impulse responses of the element groups. Two types of noise sources are considered, namely, those noise sources which are not present until the signal is applied, and those which are present for a long time before the signal is applied.

In Chapter V a general expression is obtained for the expected value of the output of the biased estimating system. Equation (25) gives this expected value as $SH_1(t)$, where S is the value of the constant being measured and $H_1(t)$ is the step response from input to output of the biased estimating system. This equation shows that the bias can always be removed from the output of the biased estimating system by dividing this output by $H_1(t)$. Thus the bias error of the biased estimating system in conjunction with the operation to remove bias is zero. In what follows it will be assumed that the biased estimating system contains the operation to remove bias.

The mean square error due to the biased estimating system for a single reading is given by equation (95), as

$$\frac{\pi}{H_1(t)^2} \sum_{i=1}^J \left[SA_1^2 \int_0^t h_1(u)^2 |K_1(t-u)| du + B_1^2 \int_0^\infty h_1(u)^2 du \right] .$$

This rather intractable expression is examined in detail in Chapter VIII where tractable results are obtained in certain special cases. An analysis is also given in this chapter which places in evidence the general effect of element group gain and bandwidth.

The more important results of this chapter are the following. Equation (105) expresses the mean square error due to the biased estimating system in terms of normalized element group responses as

$$\begin{aligned} \frac{\pi \omega_c}{H_1^*(\omega_c t)^2} \sum_{i=1}^J \frac{1}{T(1, 1-1)} \left[SA_1^2 \int_0^{\omega_c t} h_1^*(u)^2 |K_1^*(\omega_c t - u)| du \right. \\ \left. + \frac{B_1^2}{T(1, 1-1)} \int_0^\infty h_1^*(u)^2 du \right] . \end{aligned}$$

The starred variables are impulse and step responses which have been normalized with respect to gain so that the step responses approach unity for large t , and with respect to ω_c so that any normalized frequency variable, say $\bar{h}_1^*(j\omega)$, is related to the unnormalized variable by the relation

$$\bar{h}_1^*(j1) = \bar{h}_1^*(j\omega_c) .$$

The $T(u, v)$ are the gains between the u^{th} and v^{th} points in the biased estimating system. (The definitions of these variables are given in more precise detail in equations (96), (97) and (98).

In general terms if the responses h_1^* , H_1^* , K_1^* , the gains $T(1,j)$, and $\omega_c t$ are fixed, then the mean square error is directly proportional to ω_c . This is consistent with the intuitive feeling that decreasing the bandwidth of the biased estimating system will decrease the noise at the output and hence decrease the mean square error. It should be noted however that as ω_c approaches zero t must approach infinity in order for $\omega_c t$ to remain fixed. Thus "zero bandwidth" subject to the assumptions above would only apply in the steady state.

If all quantities but the gains $T(1,j)$ are held fixed, the equation above shows that ($T(1,0) = 1$ for all cases) varying the gains has no effect on the portion of the mean square error due to sources number one of both types. The contribution of other sources is decreased in proportion to the amount that the gain between the input and the position of the noise source is increased.

Consideration of the effect of the gains $T(1,j)$ on the mean square error leads to the important special case that the gain of the first element group, $T(1,1)$ is very large. For this case only the sources numbered one have appreciable effect on the mean square error. Thus the mean square error due to the biased estimating system is approximately

$$\frac{\pi\omega_c}{H_1^*(\omega_c t)^2} \left[SA_1^2 \int_0^{\omega_c t} h_1^*(u)^2 du + B_1^2 \int_0^\infty h_1^*(u)^2 du \right] .$$

Other special cases are considered in Chapter VIII.

Several bounds on the mean square error of the biased estimating system subject to various assumptions are established in Chapter VIII.

The most important of these is a lower bound on the mean square error of the biased estimating system. This bound is

$$\frac{\pi}{t}(SA_1^2 + B_1^2).$$

It applies to all linear noisy measurement systems.

As mentioned above the effect of several readings on the mean square error due to the biased estimating system is complicated. Some attention is given to this problem in Chapter VII. Perhaps representative of the results in a more complicated case are the results for two readings of the output meter. For this case it is necessary to consider not only the mean square error of the output of the biased estimating system at the times the readings are made, but also the correlation between these values. Two general statements can be made relative to the results which are presented in detail in Chapter VII. If the outputs of the biased estimating system at the times of the two readings are uncorrelated, then the mean square error is one quarter of the sum of the mean square errors at the time of each individual reading. If the mean square errors in each single reading were equal, then the effect of two readings would be to halve the mean square error. In a more general case for which correlation could exist the mean square error due to the biased estimating system is

$$\frac{\lambda(t_1, t_1) + 2\lambda(t_1, t_2) + \lambda(t_2, t_2)}{4}.$$

For such a case it is important to note that $\lambda(t_1, t_2)$ can be negative so that the mean square error could be reduced by more than the factor of two which applied for the uncorrelated case. Of course in no case is it possible for $\lambda(t_1, t_2)$ to be sufficiently negative for the mean square error to be zero.

CHAPTER X

AN EXAMPLE ILLUSTRATING THE DESIGN OF A MEASUREMENT SYSTEM

The purpose of this chapter is to consider a simple but practical measurement system to illustrate the general results of this study. The system chosen for this example, is shown in Figure 14. It represents the basic photometer circuit of several practical instruments (14), and in its practical form satisfies the restrictions imposed on the linear noisy measurement system.

The objectives of the discussion will be to first represent this system in the general framework of the linear noisy measurement system and then to use the results of the general analysis to choose that combination of system parameters which result in the smallest bias and mean square errors for a single reading of the output meter subject to either the restraint (1) the time at which the reading of the output meter is made is fixed or (2) the natural frequency of the galvanometer is fixed.

As a first step in the analysis the system can be represented by the block diagram of Figure 15. Note that the operation to remove bias has been placed after the output meter in this figure. The symbols have the following significance:

S - Constant amount of light to be measured. The light is applied to the system at $t = 0$.

$N_a(t)$ - Shot noise and radiation noise generated in the phototube referred to its input.

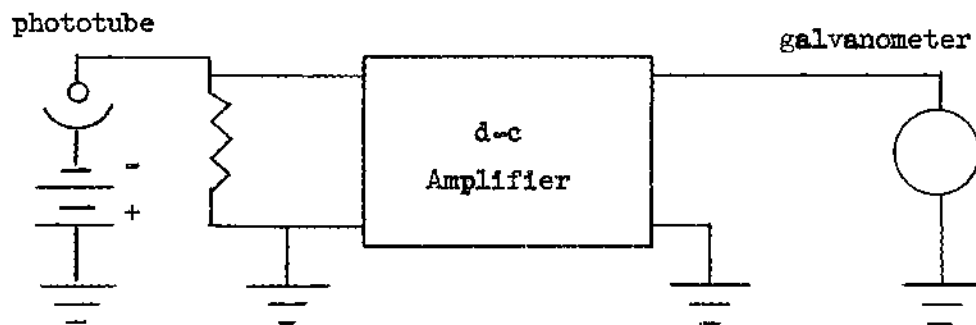


Figure 14 A Representative Measurement System

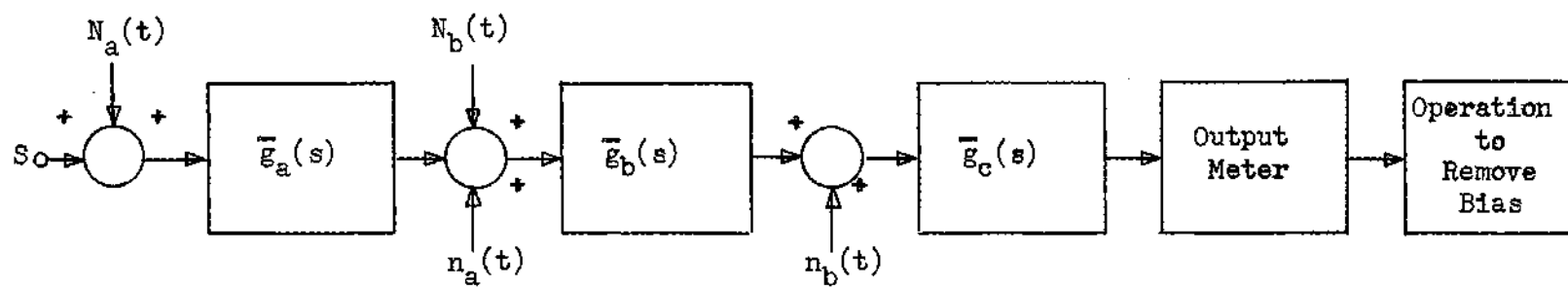


Figure 15 Block Diagram of Representative Measurement System

$N_b(t)$ - Shot noise generated in the d-c amplifier referred to its input.

$n_a(t)$ - Thermal noise in the load resistor of the phototube and the input resistor of the d-c amplifier.

$n_b(t)$ - Brownian noise and temperature noise in the galvanometer.

$\bar{g}_a(s)$ - Transfer function of the phototube and its load resistor.

$\bar{g}_b(s)$ - Transfer function of the d-c amplifier.

$\bar{g}_c(s)$ - Transfer function of the galvanometer.

It will be assumed that the galvanometer has a transfer function given by

$$\bar{g}_c(s) = \frac{\omega_n^2 K_c}{s^2 + 2\zeta_n s + \omega_n^2} , \quad (113)$$

a sensitivity from output to input denoted by V , a mean square movement error denoted by $\bar{\sigma}_m^2$ in galvanometer output units and a round-off error denoted by Γ in galvanometer output units. The phototube and d-c amplifiers will be assumed to have no frequency attenuation in the range of interest so that

$$\bar{g}_a(s) = K_a$$

and

$$\bar{g}_b(s) = K_b .$$

The power per unit bandwidth for the sources will be denoted as follows:

$$\text{source } N_a(t) = S A_a^2$$

$$\text{source } N_b(t) = S K_a A_b^2$$

$$\text{source } n_a(t) = B_a^2$$

$$\text{source } n_b(t) = B_b^2$$

The representation of this system can be easily adapted to the form and notation used for the linear noisy measurement system (see Figure 5). The resulting block diagram is given in Figure 16 where the symbols have the following significance:

$$N_1(t) = N_a(t) + \frac{N_b(t)}{K_a}$$

$$A_1^2 = A_a^2 + \frac{A_b^2}{K_a}$$

$$n_1(t) = \frac{n_a(t)}{K_a} + \frac{n_b(t)}{K_a K_b}$$

$$B_1^2 = \frac{B_a^2}{K_a^2} + \frac{B_b^2}{(K_a K_b)^2}$$

$$\bar{e}_1(s) = \bar{e}_c(s)$$

$$T(1,1) = K_a K_b K_c$$

Now the general equations for bias and mean square error are directly applicable. Thus use of equations (79) and (80) gives for bias and mean

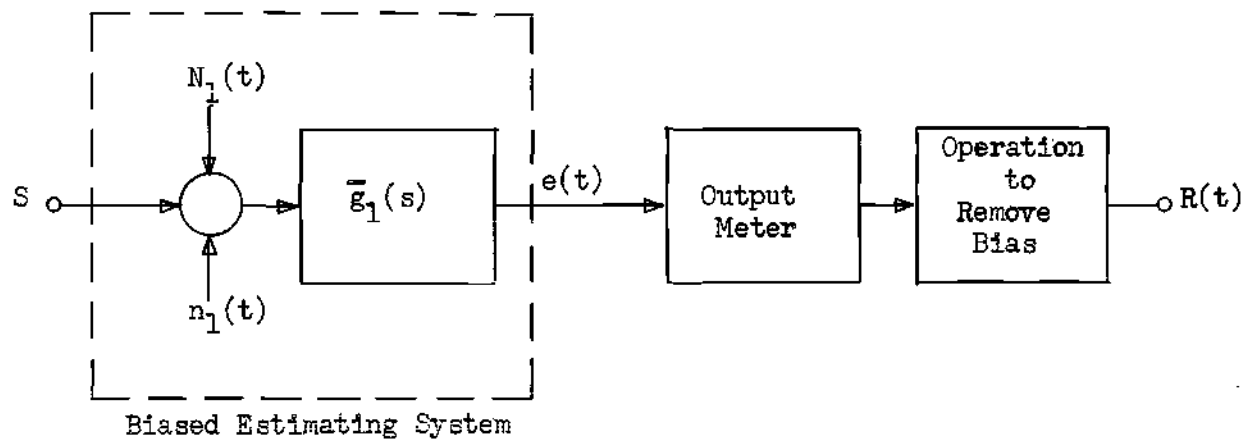


Figure 16 Representation of Measurement System in Standard Form

square error referred to the system output

$$E R(t) - S = \frac{V}{G_1(t)} E E_r \quad (114)$$

$$E [R(t) - S]^2 = E r_1(t)^2 - S^2 + \frac{V^2}{G_1(t)^2} [\bar{\sigma}_m^2 + E E_r^2] . \quad (115)$$

In this case the operation to remove bias is required to multiply the output meter output by $V/G_1(t)$. In a practical case this could be done by a time variable calibration. Conventionally the system would not be read until it reached the steady state so that calibration would be the constant $V/G_1(\infty)$.

As discussed in Chapters VII and IX it is possible to consider the design of the output meter separately from the design of the biased estimating system. Consider the design of the output meter first. The general conclusions reached in Chapter VII and summarized in Chapter IX apply specifically to this case. Thus the design considerations for the output meter are the following:

1. To minimize the bias error for a given reading, overall system gain should be adjusted so that at the time of the reading $SG_1(t)$ is as near the output meter full scale as the noise will allow. In this example either K_b or K_c could be varied to make this adjustment. Neither of these quantities affects the mean square error subject to the optimum adjustment of K_a discussed below.

2. For a fixed quantity to be measured, S , the adjustment of (1)

makes $V/G_1(t)$ essentially a constant. Thus the meter sensitivity, V , affects the value of gain required for (1), but does not affect the performance of the output meter.

3. $E E_r$ and $E E_r^2$ are bounded by $\pm \Gamma/2$, $\Gamma^2/4$ and $\Gamma^2/2$ respectively.

Thus other things being equal the choice of the galvanometer having the smallest value of Γ would minimize the effect of round-off error. A similar statement can be made relative to the mean square movement error, $\bar{\sigma}_m^2$.

Now consider the mean square error due to the biased estimating system. This component of the error can be expressed for this system in terms of the normalized response $g_1^*(t)$ as

$$E [r_1(t) - s]^2 = \frac{\pi \omega_n}{G_1^*(\omega_n t)^2} \left[s (A_a^2 + \frac{A_b^2}{K_a^2}) \int_0^{\omega_n t} g_1^*(u)^2 du + \left[\frac{B_a^2}{K_a^2} + \frac{B_b^2}{(K_a K_b)^2} \right] \int_0^{\infty} g_1^*(u)^2 du \right] \quad (116)$$

by use of equation (105). The parameters of the biased estimating system are thus t , ω_n , A_a^2 , A_b^2 , B_a^2 , B_b^2 , K_a , K_b , and ζ . Either t or ω_n is fixed depending on which of the two design problems is being considered. A design procedure which results in a choice of the parameters to minimize the mean square error of the biased estimating system will now be discussed. First, consider the gains K_a and K_b . Reference to equation (115) shows that $E [r_1(t) - s]^2$ is minimized with respect to K_a for $K_a \rightarrow \infty$. In this case

$$E (r_1(t) - S)^2 \approx \frac{\pi \omega_n}{G_1^*(\omega_n t)^2} SA_a^2 \int_0^{\omega_n t} g_1^*(u)^2 du \quad (117)$$

In most practical cases values of K_a greater than say 100 will make all of the terms containing K_a in the denominator negligible relative to the other term. Thus a choice for K_a of greater than 100 would be made to minimize $E (r_1(t) - S)^2$ relative to K_a . For this choice of K_a , $E (r_1(t) - S)^2$ is independent of K_b , A_b^2 , B_a^2 and B_b^2 . Equation (117) now gives $E (r_1(t) - S)^2$ as a function of ω_n , t and ξ , the latter parameter being contained in g_1^* and $G_1^*(\omega_n t)$. It is convenient to define $\alpha(\omega_n, \xi; t)$ by the equation

$$\alpha(\omega_n, \xi; t) = \frac{E(r_1(t) - S)^2}{\pi SA_a^2} = \omega_n \frac{\int_0^{\omega_n t} g_1^*(u)^2 du}{\left[\int_0^{\omega_n t} g_1^*(u) du \right]^2}$$

where the expression for $G_1^*(\omega_n t)$ in terms of $g_1^*(u)$ has been used. If ω_n is set equal to one, the prototype behavior referred to in Chapter VIII is obtained. Note that in this case the value of ξ determines the particular prototype.

For the purposes here it is convenient to consider the functions

$$\frac{\alpha(\omega_n, \xi; t)}{\omega_n} = \frac{\int_0^{\omega_n t} g_1^*(u)^2 du}{\left[\int_0^{\omega_n t} g_1^*(u) du \right]^2}$$

and

$$t\alpha(\omega_n, \zeta; t) = \omega_n t \frac{\int_0^{\omega_n t} g_1^*(u)^2 du}{\left[\int_0^{\omega_n t} g_1^*(u) du \right]^2}$$

Note that each of these functions is a function of $\omega_n t$ and the ratio of the two integrals involving $g_1^*(u)$.

$$\frac{\alpha(\omega_n, \zeta; t)}{\omega_n}$$

is plotted versus $\omega_n t$ for several representative values of ζ in Figure 17, and $t\alpha(\omega_n, \zeta; t)$ is plotted versus $\omega_n t$ for several values of ζ in Figure 18. The lower bound on the ordinate is plotted in each figure. Figure 18 is appropriate to choosing ζ and ω_n to give a minimum value of $\alpha(\omega_n, \zeta; t)$ if t is fixed, and Figure 17 is appropriate to choosing ζ and t to minimize $\alpha(\omega_n, \zeta; t)$ if ω_n is fixed. The procedure for making the choice of ζ and t or ζ and ω_n is clear from examination of the figures. For example if ω_n is fixed Figure 17 shows that the choice of $\zeta = 2$ and large values of t would result in the smallest values of $\alpha(\omega_n, \zeta; t)$ for the representative curves of this figure. The curve of the lower bound on $\alpha(\omega_n, \zeta; t)$ for this case shows that this value of ζ gives an actual value of $\alpha(\omega_n, \zeta; t)$ indistinguishable from its smallest possible value for the extreme value of $t = 9$ shown in the figure.

Similarly if t is fixed, examination of Figure 18 shows that roughly

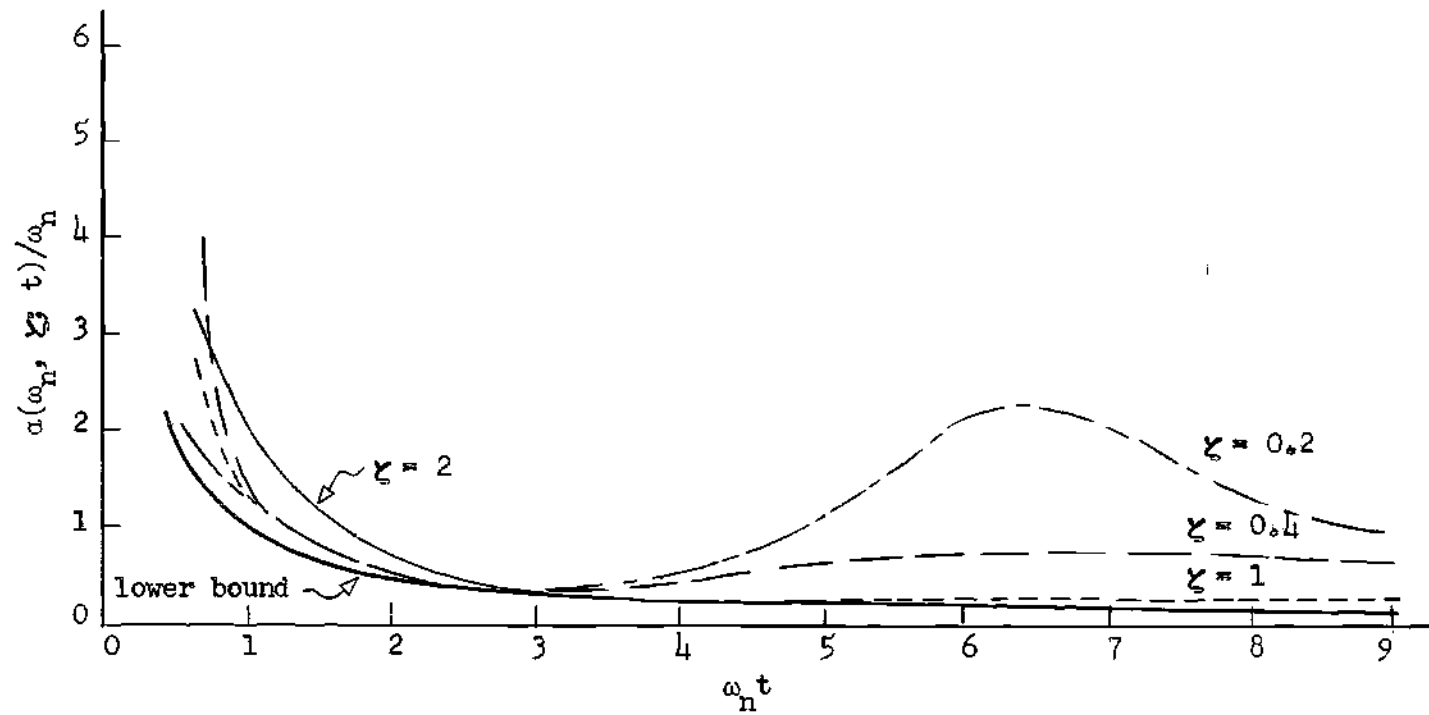


Figure 17 $a(\omega_n, \zeta; t)/\omega_n$ vs $\omega_n t$

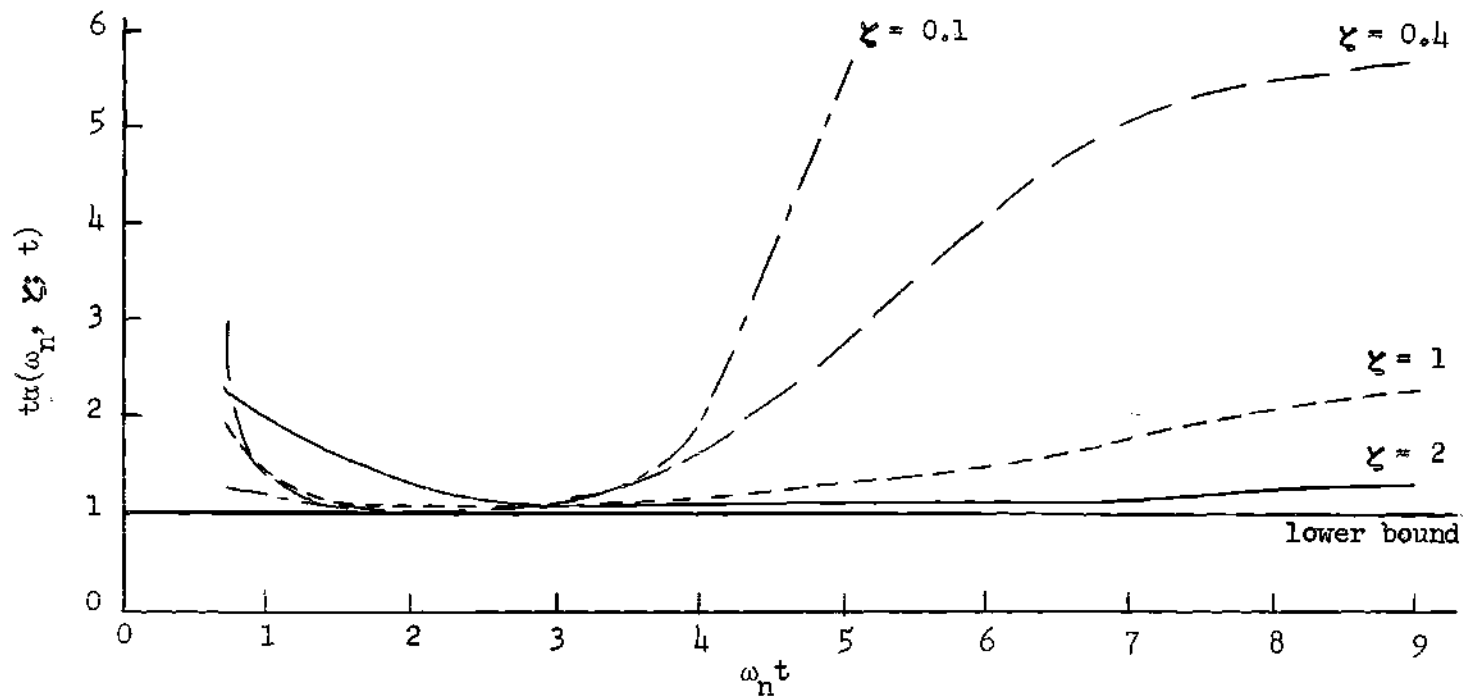


Figure 18 $ta(\omega_n, \zeta; t)$ vs $\omega_n t$

equivalent results can be obtained over a range of ω_n from $1.5/t$ to $7/t$. In this range, the values 0.2, 0.4 and 1 for ξ give comparable results for ω_n in the range $1.5/t$ to $3/t$, while $\xi = 2$ is the best choice for ω_n in the range $3/t$ to $7/t$. Again the actual value of $\alpha(\omega_n, \xi; t)$ resulting from these choices of ω_n and ξ are close to the value of the lower bound on $\alpha(\omega_n, \xi; t)$.

It is clear that other design problems such as those resulting from the condition that both ω_n and t are fixed or that both ω_n and t are arbitrary could be handled using these, or similar, design curves. In the former case it is interesting to note from Figure 17 that if $\omega_n t$ is fixed at less than say 3, $\xi = 0.2$ results in a value of $\alpha(\omega_n, \xi; t)$ smaller than those for higher values of ξ . In fact the curve for $\xi = 0.2$ is close to the lower bound in the range $\omega_n t$ from say 1 to 3.

Once values for ω_n and t have been chosen so that $\alpha(\omega_n, \xi; t)$ is fixed, equation (117) shows that the actual mean square error due to the biased measurement system is given by

$$E (r_1(t) - S)^2 = \pi S A_a^2 \alpha(\omega_n, \xi; t). \quad (118)$$

Thus it is clear that small values of A_a^2 result in the smallest error of this type. The effect of the explicit appearance of S in the above equation has been discussed in Chapter III. As indicated in that chapter, S is unknown but will be estimated by the measurement. Use of the estimated value of S can give only an approximate value for $E(r_1(t) - S)^2$ which will have the effect of adding some uncertainty to the final probability statement which summarizes the result of the measurement. Finally it should be noted that reducing S , which has the effect of reducing $E(r_1(t) - S)^2$,

is not necessarily the proper thing to do. For this consideration fractional mean square error is another measure of system performance which should be evaluated. In this case fractional mean square error is given by

$$\text{fractional mean square error} = \frac{\pi A_a^2}{S} \alpha(\omega_n, \xi; t) \quad (119)$$

This equation shows that fractional mean square error is inversely proportional to S . Thus if it is possible to control the approximate size of S , a large value would be chosen in most cases.

The results of this example will now be summarized. The objective of the design considerations is to minimize bias and mean square errors. The discussion is limited to a single reading of the output meter, while the system itself is subject to either the restraint (1) the time at which the reading of the output meter is made is fixed or (2) the natural frequency of the galvanometer is fixed. Subject to these conditions and restraints the parameter values which minimize bias and mean square error are the following:

1. Γ and σ_m and A_a^2 should be as small as possible.
2. The gain K_a should be large, say greater than 100.
3. K_b or K_c should be adjusted so that for each reading $SG_1(t)$ is as close to full scale on the output meter as the noise will allow
4. The meter sensitivity, V , affects the particular choice of gain in (3), but subject to the adjustment of (3), bias and mean square errors are essentially independent of K_b , K_c and V .
5. The choice given in (2) for K_a makes bias and mean square error substantially independent of A_b^2 , B_a^2 and B_b^2 .

6. If ω_n is fixed, large values of ξ and large values of the time of the reading, t , give the smallest mean square error. For the extreme values of $\xi = 2$ and $t = 9$ given in the representative curves of Figure 17, the actual mean square error cannot be distinguished from the lower bound.

7. If t is fixed a range of ω_n from approximately $1.5/t$ to $7/t$ gives comparable results. The choice for ξ depends on which values of ω_n are chosen. For example if ω_n is in the range $1.5/t$ to $3/t$ the values $\xi = 0.2, 0.4$ and 1 for ξ give comparable results, while $\xi = 2$ is a better choice if ω_n is in the range $3/t$ to $7/t$. For any of these choices the difference between the actual mean square error and the lower bound does not exceed 20%.

8. If both ω_n and t are fixed and if $\omega_n t$ is less than say 3, a value of $\xi = 0.2$ gives the smallest mean square error for the representative values used in Figure 17.

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VITA

Joseph L. Hammond, Jr., was born October 16, 1927 in Birmingham, Alabama. He was graduated from the public schools in Birmingham in 1945. After this he served with the U. S. Navy until his discharge in 1946. Mr. Hammond attended Auburn University for the 1946-47 academic year, and Howard College for the 1947-48 academic year. In 1948 he transferred to the Massachusetts Institute of Technology where he received the BS and MS degrees in 1952.

After graduation from M.I.T. Mr. Hammond was employed until 1955 by the Southern Research Institute in Birmingham, Alabama as a Research Engineer. In 1955 he came to the Georgia Institute of Technology as Assistant Professor of Electrical Engineering and worked toward his Ph.D. During this period, in addition to working toward his doctorate, Mr. Hammond worked part-time for the Engineering Experiment Station and on several occasions as a consultant to industrial firms.

He is a member of the Tau Beta Pi national honorary engineering fraternity and of the Institute of Radio Engineers. He is married to the former Edith Van Keuren of Birmingham, Alabama, and has three children.